PARTIAL AND SPECTRAL-VISCOSITY MODELS FOR GEOPHYSICAL FLOWS

QINGSHAN CHEN, MAX GUNZBURGER AND XIAOMING WANG

Dedicated to Roger Temam on the occasion of his seventieth birthday

ABSTRACT. Two models based on the hydrostatic primitive equations are proposed. The first model is the primitive equations with *partial* viscosity *only*, and is oriented towards large-scale wave structures in the ocean and atmosphere. The second model is the viscous primitive equations with spectral eddy viscosity, and is oriented towards turbulent geophysical flows. For both models, the existence and uniqueness of global strong solutions is established. For the second model, the convergence of the solutions to the solutions of the classical primitive equations as eddy viscosity parameters tend to zero is also established.

1. INTRODUCTION

We study two models for geophysical flows based on the hydrostatic primitive equations; both are designed to faithfully simulate certain phenomena in the geophysical flows but they are motivated by different physical considerations. A distinctive characteristic of the flows under consideration is that the vertical scale ($\sim$ 10km) is much smaller than the horizontal scale ($\sim$ 6000km). Thanks to this disparity, a hydrostatic approximation is possible, and gives rise of the primitive-equations.

On the mathematical side, the theory for the primitive equations is fairly complete; see, e.g., the pioneering work in [22, 23], and the survey article [28]. In particular, in contrast with the Navier-Stokes equations [9, 33], the primitive equations have been shown to have unique global strong solutions [7, 20, 21, 18].

The first model we study aims to faithfully simulate large-scale coherent structures including wave phenomena in the ocean and atmosphere. For relevant discussions of this topic, see, e.g., [12, 24, 27].

Date: October 22, 2010.
This article appeared in *Chinese Annal of Mathematics Ser. B*, 31:579-606, 2010.
DOI: 10.1007/s11401-010-0607-2.
Corresponding author: Qingshan Chen (qchen3@fsu.edu).

For the phenomena that we are interested in, both the ocean and atmosphere are close to being inviscid. Therefore, the inviscid primitive equation is the preferred model. However, it is very costly to simulate the inviscid primitive equations directly due to the small scales embodied within the model. What we propose here is a new model with eddy viscosity added to the small scale (high frequency) part only while keeping the large scale (low frequency) features intact (at least directly). Intuitively, this type of model would reduce the complexity due to the damping on small scales whereas keeping the desirable large scale structures. Such a naive approach may not work all the time due to the cascade of energy induced by the nonlinear advection term. However, we can easily find situations where such cascade is small or negligible. Indeed, it is easy to find exact large-scale solutions to this primitive equation with partial viscosity; see below. Of course, the existence of such examples do not fully justify the model, and extensive numerical experiments are called for which is our future plan. The idea of partial damping on the high frequency components of the system is not alien to the geophysical community (see, e.g., [11, 25]) or the mathematical community (see e.g. [26, 4, 5, 10]), although the application to the primitive equations is new here. Our goal in this paper is to demonstrate the global well posedness of this model with partial viscosity.

The second model is oriented at turbulence modeling for geophysical flows. The simulation of three-dimensional turbulent flows is a formidable task due to the need to resolve the small scale fluctuations or *eddies* that have subtle effects on the large-scale dynamics of the flow. To make this problem computationally tractable, these effects must be modeled. In one approach, the velocity field is averaged over a small radius to derive equations in terms of the averaged velocity. For nonlinear equations, there arises the problem of *closure* because the product operation is not closed under the averaging process. To obtain a closed system, the average of the nonlinear terms in the equations must be approximated and expressed solely in terms of averaged quantities. The way in which this is done gives rise to a variety of models. The approach we consider, called the *eddy-viscosity method*, treats the Reynolds stress as a viscous effect caused by the transport and dissipation of energy due to the small-scale eddies. For this reason, this additional viscosity is called the *eddy viscosity* or *turbulent viscosity*. The turbulence model of Smagorinsky [31] belongs to this type. For an overall survey on issues related to these models, see [3].

Unfortunately, a straightforward application of the approach described above leads to the over smearing of the large-scale structures

in the fluid. To remedy this unwanted effect, it has been proposed that the eddy viscosity be added only to the *subgrid scales*. In this way, one hopes to prevent the large-scale structure from being smeared away. Here, we examine a particular class of models of this type called *spectral-viscosity* or *spectral-vanishing-viscosity* models, in which the scales are defined in terms of Fourier modes. The subgrid viscosity is simply realized as an addition of the artificial viscosity only to the high-frequency modes. The most intuitive way of doing this to insert a high-pass filter to the standard eddy viscosity. This approach was considered in [13] for hyperviscosity on the Navier-Stokes equations and in [15] for nonlinear as well as hyper-viscosity on the Navier Stokes Equations. In both works, the well-posedness of the resulting spectral viscosity is proven, and the consistency of these model with the original Navier-Stokes equations is discussed.

We employ the idea of *spectral viscosity* to build and analyze a turbulence model for the geophysical flows in the ocean and atmosphere. As mentioned above, the primitive equations, even without any eddy viscosity, have been shown to have unique global solutions, provided the initial and boundary data are sufficiently smooth. For our model, we prove its global well posedness, which should not come as a surprise. In addition, we will show the convergence of the solutions of the model to the solutions of the primitive equations without any eddy viscosity, as the eddy viscosity parameters tend to zero. This is not possible yet for the Navier-Stokes equations because there convergence is only shown to be in a weak sense.

We should point out that this technique, usually under the name of *spectral viscosity* or *spectral-vanishing viscosity*, is known in terms of turbulence modelling (see [19, 13, 2, 15, 32], and below), and to applications in geophysical fluid dynamics [11]. However, to the best of our knowledge, the well-posedness result for the three-dimensional nonlinear primitive equations with partial viscosity is new.

The paper is organized as follows. In Section 2, we introduce and prove the well posedness of a model with only partial viscosity. In Section 3, we introduce the linear spectral eddy-viscosity model. In Section 3.2, we prove the existence and uniqueness of strong solutions to the model. In Section 3.3 we study the convergence of the solutions of the model to the solutions of the original primitive equations.

2. A model with partial viscosity

In this section we study a model with partial high–frequency viscosity only. The lower modes are not damped directly. This feature renders

the model suitable for large scale coherent structures in the ocean and atmosphere, because non–physical large scale damping could change the large scale coherent structures over time.

The model reads

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + w\frac{\partial \mathbf{u}}{\partial z} + f\mathbf{k}\times\mathbf{u} + \frac{1}{\rho_0}\nabla p - \mu\Delta(I-P_{M,N})\mathbf{u} - \nu\frac{\partial^2}{\partial z^2}(I-P_{M,N})\mathbf{u} = \mathbf{F}, \tag{2.1}$$

$$\frac{\partial p}{\partial z} = -\rho_0 g, \tag{2.2}$$

$$\nabla\cdot\mathbf{u} + \frac{\partial w}{\partial z} = 0, \tag{2.3}$$

In the above, $\mathbf{u} = (u, v)$ is the horizontal velocity, w is the vertical velocity, μ and ν are the horizontal and vertical kinetic viscosities, respectively. We use ∇ and Δ to denote the 2D horizontal gradient and Laplacian operators, respectively. The operator $(I - P_{M,N})$ represents the high–pass filter and will be defined later on.

We consider a rectangular domain $\Omega = M \times (-H, 0)$, with $M = (0, L_x)\times(0, L_y)$. We consider periodic boundary conditions in both the x and y directions, and free–slip, non–penetration boundary conditions in the vertical directions. More precisely,

$$\mathbf{u}(x,y,z,t) = \mathbf{u}(x+L_x,y,z,t), \tag{2.4}$$

$$\mathbf{u}(x,y,z,t) = \mathbf{u}(x,y+L_y,z,t), \tag{2.5}$$

$$\frac{\partial \mathbf{u}}{\partial n}|_{z=0} = \frac{\partial \mathbf{u}}{\partial n}|_{z=-H} = 0, \tag{2.6}$$

$$w|_{z=0} = w|_{z=-H} = 0, \tag{2.7}$$

$$p|_{z=0} = p_0. \tag{2.8}$$

Under the settings just described, we can define the high–pass filter in terms of Fourier frequencies. Specifically, for each function $\mathbf{u} \in (L^2(\Omega))^2$ we let

$$P_{M,N}\mathbf{u} = \sum_{|\mathbf{m}|_{sup}\le M\,\&\,n\le N} \widehat{\mathbf{u}}_{\mathbf{m},n} e^{i\mathbf{m}\cdot\mathbf{x}'}\cos nz' \tag{2.9}$$

where

$$\begin{aligned}
&\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2,\\
&|\mathbf{m}|_{\sup} = \max(|m_1|, |m_2|),\\
&\mathbf{x}' = 2\pi(\frac{x}{L_x}, \frac{y}{L_y}),\\
&z' = \frac{\pi z}{H},\\
&\widehat{\mathbf{u}}_{\mathbf{m},n} = \int_\Omega \mathbf{u}(x, y, z) e^{-i\mathbf{m}\cdot\mathbf{x}'} \cos nz' \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}$$

Remark 2.1. The issue of suitable physical boundary conditions for the inviscid primitive equations is an unresolved one. Partial results concerning the linearized primitive equations are available in a series of papers [29], [8], and [30]. There, an infinite set of nonlocal boundary conditions were proposed, which guaranteed the well–posedness of the linearized system. Here we avoid this issue and use the periodic boundary conditions on the lateral boundaries.

Remark 2.2. As we have touched upon in Introduction, we can easily identify some large–scale motions which are exact solutions of the inviscid primitive equations. For example, let

$$\psi = \sin\left(\frac{4\pi x}{L_1}\right)\cos\left(\frac{2\pi y}{L_2}\right),$$

and

$$\mathbf{u} = \nabla^\perp \psi.$$

With the surface pressure p_0 given by

$$p_0 = f\psi + \frac{2\pi^2}{L_2^2}\cos\left(\frac{4\pi x}{L_1}\right)^2 - \frac{8\pi^2}{L_1^2}\cos\left(\frac{2\pi y}{L_2}\right)^2,$$

$(\mathbf{u}, 0)$ is a set of exact solutions of the inviscid primitive equations, that is, the system (2.1)–(2.8) without the viscosities, or with partial viscosity with $M > 4$.

2.1. **The barotropic and baroclinic modes.** As usual, p and w can be expressed in terms of $\mathbf{u}$. Specifically, integrating (2.2) from z to 0, and using the boundary condition (2.8) we obtain

$$p(x, y, z, t) = p_0(x, y, t) - \rho_0 g z, \tag{2.10}$$

Integrating (2.3) from z to 0, and using the boundary condition (2.7), we obtain

$$w(x, y, z, t) = \int_z^0 \nabla\cdot\mathbf{u}\,\mathrm{d}\xi = \nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}\xi. \tag{2.11}$$

Setting $z = -H$ in (2.11), and using (2.7) again, we find

$$\nabla \cdot \int_{-H}^{0} \mathbf{u} \, \mathrm{d}z = 0. \tag{2.12}$$

We substitute (2.10) and (2.11) into (2.1), and obtain a single closed equation for $\mathbf{u}$:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + (\nabla \cdot \int_{z}^{0} \mathbf{u} \, \mathrm{d}\xi)\frac{\partial \mathbf{u}}{\partial z} + f\mathbf{k} \times \mathbf{u} + \frac{1}{\rho_0}\nabla p_0 - \\ \mu\Delta(I - P_{M,N})\mathbf{u} - \nu\frac{\partial^2}{\partial z^2}(I - P_{M,N})\mathbf{u} = \mathbf{F}. \tag{2.13}$$

It turns out essential to rewrite this equation and work with the following form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \left(\nabla \cdot \int_{z}^{0} \mathbf{u} \, \mathrm{d}\xi\right)\frac{\partial \mathbf{u}}{\partial z} + f\mathbf{k} \times \mathbf{u} + \frac{1}{\rho_0}\nabla p_0 - \\ \mu\Delta\mathbf{u} - \nu\frac{\partial^2}{\partial z^2}\mathbf{u} = \mathbf{F} - \mu\Delta P_{M,N}\mathbf{u} - \nu\frac{\partial^2}{\partial z^2}P_{M,N}\mathbf{u}. \tag{2.14}$$

We note here that $P_{M,N}\mathbf{u}$ contains only finite number modes of $\mathbf{u}$. The prognostic variable $\mathbf{u}$ satisfies the equation (2.14), the constraint (2.12), and the boundary conditions (2.4)–(2.6). To complete the system, we also require $\mathbf{u}$ to satisfy the following initial condition:

$$\mathbf{u}(x, y, z, 0) = \mathbf{u}_0(x, y, z). \tag{2.15}$$

The diagnostic variables w and p are given by (2.11) and (2.10) respectively.

We now specify the barotropic and baroclinic modes of equation (2.14). We let

$$\overline{\mathbf{u}}(x, y, t) = \frac{1}{H}\int_{-H}^{0} \mathbf{u}(x, y, z, t) \, \mathrm{d}z, \tag{2.16}$$

which denotes the barotropic mode of the primitive variables. We also denote by

$$\mathbf{u}'(x, y, z, t) = \mathbf{u}(x, y, z, t) - \overline{\mathbf{u}}(x, y, t) \tag{2.17}$$

the baroclinic mode of the primitive variables. It is easy to see that

$$\overline{\mathbf{u}'} = 0. \tag{2.18}$$

From (2.4) and (2.5) we derive the boundary conditions for $\overline{\mathbf{u}}$:

$$\overline{\mathbf{u}}(x, y, t) = \overline{\mathbf{u}}(x + L_x, y, t), \tag{2.19}$$

$$\overline{\mathbf{u}}(x, y, t) = \overline{\mathbf{u}}(x, y + L_y, t). \tag{2.20}$$

By (2.12) we see that $\overline{\mathbf{u}}$ satisfies the following constraint:

$$\nabla \cdot \overline{\mathbf{u}} = 0. \tag{2.21}$$

Then $\mathbf{u}'$ also satisfy the periodic boundary conditions on the lateral boundary,

$$\mathbf{u}'(x, y, z, t) = \mathbf{u}'(x + L_x, y, z, t), \tag{2.22}$$

$$\mathbf{u}'(x, y, z, t) = \mathbf{u}'(x, y + L_y, z, t). \tag{2.23}$$

It inherits the boundary conditions for $\mathbf{u}$ on the top and bottom,

$$\left.\frac{\partial \mathbf{u}'}{\partial z}\right|_{z=0} = \left.\frac{\partial \mathbf{u}'}{\partial z}\right|_{z=-H} = 0. \tag{2.24}$$

We now derive the equations that $\overline{\mathbf{u}}$ and $\mathbf{u}'$ satisfy by first taking average of the equation (2.14):

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \overline{(\mathbf{u} \cdot \nabla)\mathbf{u}} + \overline{w\frac{\partial \mathbf{u}}{\partial z}} + f\mathbf{k} \times \overline{\mathbf{u}} + \frac{1}{\rho_0}\nabla p_0 - \mu\Delta\overline{\mathbf{u}} - \nu\overline{\frac{\partial^2 \mathbf{u}}{\partial z^2}} = \overline{\mathbf{F}} - \mu\overline{\Delta P_{M,N}\mathbf{u}} - \nu\overline{\frac{\partial^2}{\partial z^2}P_{M,N}\mathbf{u}}. \tag{2.25}$$

We notice that

$$\overline{(\frac{\partial^2 \mathbf{u}}{\partial z^2})} = \frac{1}{H}\int_{-H}^{0} \frac{\partial^2 \mathbf{u}}{\partial z^2}\,\mathrm{d}z = \frac{1}{H}\left.\frac{\partial \mathbf{u}}{\partial z}\right|_{-H}^{0} = 0, \quad (\text{ by } (2.6)),$$

$$\overline{\frac{\partial^2}{\partial z^2}P_{M,N}\mathbf{u}} = \frac{1}{H}\int_{-H}^{0} \frac{\partial^2}{\partial z^2}P_{M,N}\mathbf{u}\,\mathrm{d}z = 0,$$

$$\overline{\Delta P_{M,N}\mathbf{u}} = \Delta P_{M,N}\overline{\mathbf{u}}.$$

By using (2.18), we find that

$$\overline{(\mathbf{u} \cdot \nabla)\mathbf{u}} = (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}} + \overline{(\mathbf{u}' \cdot \nabla)\mathbf{u}'}.$$

Using (2.12), (2.18) and (2.21), we find

$$\overline{w\frac{\partial \mathbf{u}}{\partial z}} = \overline{(\nabla \cdot \mathbf{u}') \cdot \mathbf{u}'}.$$

Hence the equation for $\overline{\mathbf{u}}$ can be written as

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}} + [\overline{(\mathbf{u}' \cdot \nabla)\mathbf{u}'} + \overline{(\nabla \cdot \mathbf{u}')\mathbf{u}'}] + f\mathbf{k} \times \overline{\mathbf{u}} + \frac{1}{\rho_0}\nabla p_0 - \mu\Delta\overline{\mathbf{u}} = \overline{\mathbf{F}} - \mu\Delta P_{M,N}\overline{\mathbf{u}}. \tag{2.26}$$

The barotropic variable $\overline{\mathbf{u}}$ satisfies the following conditions:

$$\overline{\mathbf{u}}(x,y,t) = \overline{\mathbf{u}}(x+L_x,y,t), \tag{2.27}$$
$$\overline{\mathbf{u}}(x,y,t) = \overline{\mathbf{u}}(x,y+L_y,t). \tag{2.28}$$
$$\nabla\cdot\overline{\mathbf{u}} = 0. \tag{2.29}$$

Subtracting (2.26) from (2.14) we obtain the equation for the baroclinic mode:

$$\begin{aligned}\frac{\partial\mathbf{u}'}{\partial t} + (\mathbf{u}'\cdot\nabla)\mathbf{u}' + \left(\nabla\cdot\int_z^0 \mathbf{u}'\,\mathrm{d}\xi\right)\frac{\partial\mathbf{u}'}{\partial z} + f\mathbf{k}\times\mathbf{u}' - \mu\Delta\mathbf{u}' - \nu\frac{\partial^2\mathbf{u}'}{\partial z^2}\\ + \left[(\mathbf{u}'\cdot\nabla)\overline{\mathbf{u}} + (\overline{\mathbf{u}}\cdot\nabla)\mathbf{u}' - \left(\overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}'} + \overline{(\nabla\cdot\mathbf{u}')\mathbf{u}'}\right)\right] = \mathbf{F}' -\\ \mu\Delta P_{M,N}\mathbf{u}' - \nu\frac{\partial^2}{\partial z^2}P_{M,N}\mathbf{u}'.\end{aligned} \tag{2.30}$$

In addition $\mathbf{u}'$ satisfy

$$\mathbf{u}'(x,y,t) = \mathbf{u}'(x+L_x,y,t), \tag{2.31}$$
$$\mathbf{u}'(x,y,t) = \mathbf{u}'(x,y+L_y,t), \tag{2.32}$$
$$\left.\frac{\partial\mathbf{u}'}{\partial z}\right|_{z=0} = \left.\frac{\partial\mathbf{u}'}{\partial z}\right|_{z=-H} = 0, \tag{2.33}$$
$$\int_{-H}^0 \mathbf{u}'\,\mathrm{d}z = 0. \tag{2.34}$$

2.2. **Global well–posedness of the model with partial viscosity.** In order to have uniqueness for the solution of (2.1)–(2.8) we work with functions that have zero average over Ω. Indeed it can be checked that if the initial data and the forcing have zero average over Ω, then the solutions have zero average over Ω at any time.

We use the convention $\dot{L}^2(\Omega)$, $\dot{H}^1(\Omega)$, etc. to denote function spaces that have zero average over Ω. We let

$$\begin{aligned} H &= (\dot{L}^2(\Omega))^2,\\ \mathcal{V} &= \big\{\mathbf{u}\in\dot{\mathcal{C}}^\infty(R^3)^2 \,|\, \mathbf{u} \text{ periodic in } x \text{ with period } L_x, \text{ periodic in } y \text{ with}\\ &\qquad \text{period } L_y, \text{ periodic and even in } z \text{ ith period } 2H\big\},\\ V &= \overline{\mathcal{V}}^{H^1}(\text{ closure of } \mathcal{V} \text{ in } (H^1(\Omega))^2).\end{aligned}$$

The inner product and norm of H will be denoted as $(\cdot,\,\cdot)$ and $|\cdot|$, respectively. The space V inherits the inner product and norm of H^1, which will be denoted as $((\cdot,\,\cdot))$ and $||\cdot||$, respectively.

Since the functions in V has zero spatial averages, we have the following Poincaré inequality for functions in V:

$$|\mathbf{u}|^2 \leq C(|\nabla \mathbf{u}|^2 + |\frac{\partial \mathbf{u}}{\partial z}|^2). \tag{2.35}$$

Therefore $(|\nabla \mathbf{u}|^2 + |\frac{\partial \mathbf{u}}{\partial z}|^2)^{\frac{1}{2}}$ is equivalent to the usual H^1 norm, and can be taken as the norm for V.

In what follows we abuse the notation by denoting every generic constant by C. Such constants may depend on the domain Ω and the function spaces in the context, but we omit such dependence in the notation. But if the constant depends on any other parameters, such as M, N etc., we shall use a specific symbol and specify such dependence in the notation.

We shall prove the global existence and uniqueness of strong solutions to the system (2.1)–(2.8).

Theorem 2.1. *For a given $T > 0$, let $\mathbf{F} \in L^2(0, T; H)$, $\mathbf{u}_0 \in V$. Then there exists a unique strong solution $\mathbf{u} \in \mathcal{C}([0, T]; V) \cap L^2(0, T; H^2(\Omega))$ of the system* (2.1)–(2.8) *which depends continuously on the initial data.*

We shall first obtain some key estimates that will be needed for the proof of Theorem 2.1. The proof of the theorem will be furnished at the end of this subsection.

L^2 estimates
We multiply (2.14) by $\mathbf{u}$, integrate by parts over Ω, using the boundary conditions (2.4)–(2.8), we obtain

$$\begin{aligned}(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{u}) + ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{u}) + \left((\nabla \cdot \int_z^0 \mathbf{u}\, \mathrm{d}\xi)\frac{\partial \mathbf{u}}{\partial z},\, \mathbf{u}\right) + \\ (f\mathbf{k} \times \mathbf{u}, \mathbf{u}) + \frac{1}{\rho_0}(\nabla p_0, \mathbf{u}) + \mu|\nabla \mathbf{u}|^2 + \nu \left|\frac{\partial \mathbf{u}}{\partial z}\right|^2 \\ = (\mathbf{F}, \mathbf{u}) + \mu_\delta |\nabla P_{M,N}\mathbf{u}|^2 + \nu_\delta \left|\frac{\partial}{\partial z} P_{M,N}\mathbf{u}\right|^2.\end{aligned} \tag{2.36}$$

We notice that

$$\left(\frac{\partial \mathbf{u}}{\partial t},\, \mathbf{u}\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|^2, \tag{2.37}$$

$$(\nabla p_0,\, \mathbf{u})_\Omega = H(\nabla p_0,\, \overline{\mathbf{u}}) = 0, \tag{2.38}$$

$$(f\mathbf{k} \times \mathbf{u}, \mathbf{u}) = 0. \tag{2.39}$$

Let

$$b(\mathbf{u}, \widetilde{\mathbf{u}}, \mathbf{u}^{\#}) = \left((\mathbf{u} \cdot \nabla \widetilde{\mathbf{u}}) + w(\mathbf{u}) \frac{\partial \widetilde{\mathbf{u}}}{\partial z},\ \mathbf{u}^{\#} \right), \tag{2.40}$$

where $w(\mathbf{u})$ is defined as in (2.11). We can verify that the trilinear operator $b(\mathbf{u}, \widetilde{\mathbf{u}}, \mathbf{u}^{\#})$ is skew symmetric with respect to the last two arguments, that is,

$$b(\mathbf{u}, \widetilde{\mathbf{u}}, \mathbf{u}^{\#}) = -b(\mathbf{u}, \mathbf{u}^{\#}, \widetilde{\mathbf{u}}). \tag{2.41}$$

Then it is inferred from (2.41) that

$$b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0. \tag{2.42}$$

By (2.36), (2.37), (2.38), (2.39) and (2.42) we have

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |\mathbf{u}|^2 + \mu |\nabla \mathbf{u}|^2 + \nu |\frac{\partial}{\partial z} \mathbf{u}|^2 = (\mathbf{F}, \mathbf{u}) + \\ \mu |\nabla P_{M,N} \mathbf{u}|^2 + \nu |\frac{\partial}{\partial z} P_{M,N} \mathbf{u}|^2. \end{aligned} \tag{2.43}$$

Since $P_{M,N}\mathbf{u}$ contains only finite number of Fourier modes,$|\nabla P_{M,N}\mathbf{u}|^2$, as well as $|\frac{\partial}{\partial z} P_{M,N}\mathbf{u}|^2$ can be bounded by $|\mathbf{u}|^2$. More generally, for functions in $H^k(\Omega)$, there exists a constant $C_k(M, N, \Omega)$, which is independent of the function, such that

$$|P_{M,N}\mathbf{u}|_{H^k} \leq C_k |\mathbf{u}|_{L^2}. \tag{2.44}$$

Using (2.44) and the Cauchy–Schwarz inequality, we derive from (2.43) that

$$\frac{\mathrm{d}}{\mathrm{d}t} |\mathbf{u}|^2 + \mu |\nabla \mathbf{u}|^2 + \nu |\frac{\partial}{\partial z} \mathbf{u}|^2 \leq C |\mathbf{F}|^2 + C_1 |\mathbf{u}|^2. \tag{2.45}$$

Applying the Gronwall inequality to (2.45) yields

$$|\mathbf{u}(\cdot, t)|^2 + \mu \int_0^t |\nabla \mathbf{u}|^2 \, \mathrm{d}s + \nu \int_0^t \left| \frac{\partial}{\partial z} \mathbf{u} \right|^2 \, \mathrm{d}s \leq J_1(t), \tag{2.46}$$

where

$$J_1(t) \equiv e^{C_1 t} \left(|u_0|^2 + C \int_0^t |\mathbf{F}|^2 \, \mathrm{d}s \right). \tag{2.47}$$

L^6 estimate on $\mathbf{u}'$

We take inner product of (2.30) with $|\mathbf{u}'|^4 \mathbf{u}'$, and integrate by parts

over Ω to obtain

$$\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}'|^6_{L^6} + b(\mathbf{u}', \mathbf{u}', |\mathbf{u}'|^4\mathbf{u}') + \int_\Omega f\mathbf{k} \times \mathbf{u}' \cdot |\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega + \int_\Omega \Big((\mathbf{u}' \cdot \nabla)\overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}' - \overline{(\mathbf{u}' \cdot \nabla)\mathbf{u}' + (\nabla \cdot \mathbf{u}')\mathbf{u}'}\Big)\,|\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega - \int_\Omega \left(\mu\Delta\mathbf{u}' + \nu\frac{\partial^2\mathbf{u}'}{\partial z^2}\right)|\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega = \int_\Omega \mathbf{F}' \cdot |\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega - \int_\Omega \left(\mu\Delta P_{M,N}\mathbf{u}' + \nu\frac{\partial^2}{\partial z^2}P_{M,N}\mathbf{u}'\right)|\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega \quad (2.48)$$

In the above, the trilinear operator $b(\cdot,\ \cdot,\ \cdot)$ is defined as in (2.40). We can verify by calculations that

$$b(\mathbf{u}', \mathbf{u}', |\mathbf{u}'|^4\mathbf{u}') = -2b(\mathbf{u}', \mathbf{u}', |\mathbf{u}'|^4\mathbf{u}'), \quad (2.49)$$

which implies that

$$b(\mathbf{u}', \mathbf{u}', |\mathbf{u}'|^4\mathbf{u}') = 0. \quad (2.50)$$

Since $k \times \mathbf{u}'$ is orthogonal to $\mathbf{u}'$, we have

$$\int_\Omega f\mathbf{k} \times \mathbf{u}' \cdot |\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega = 0. \quad (2.51)$$

Noticing the divergence free condition (2.29) for $\overline{\mathbf{u}}$ and the horizontal periodic boundary conditions (2.27)–(2.28) and (2.31)–(2.32) for $\overline{\mathbf{u}}$ and $\mathbf{u}'$ respectively, we find

$$\int_\Omega (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}' \cdot |\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega = 0. \quad (2.52)$$

For the inner products involving the diffusion terms, we find

$$-\int_\Omega \mu\Delta\mathbf{u}' \cdot |\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega = \mu\int_\Omega |\nabla\mathbf{u}'|^2|\mathbf{u}'|^4\,\mathrm{d}\Omega + \mu\int_\Omega \left|\nabla|\mathbf{u}'|^2\right|^2|\mathbf{u}'|^2\,\mathrm{d}\Omega, \quad (2.53)$$

$$-\int_\Omega \nu\frac{\partial^2\mathbf{u}'}{\partial z^2} \cdot |\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega = \nu\int_\Omega |\frac{\partial\mathbf{u}'}{\partial z}|^2|\mathbf{u}'|^4\,\mathrm{d}\Omega + \nu\int_\Omega \left|\frac{\partial}{\partial z}|\mathbf{u}'|^2\right|^2|\mathbf{u}'|^2\,\mathrm{d}\Omega. \quad (2.54)$$

For integrals on the right–hand side of (2.48), we use (2.44) (with $k = 2$) to find that

$$\begin{aligned}
&\int_\Omega -\mu\Delta P_{M,N}\mathbf{u}' \cdot |\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega \\
&\le \mu|\Delta P_{M,N}\mathbf{u}'|_{L^2}|\mathbf{u}'|_{L^{10}}^5 \\
&\le C_2|\mathbf{u}'|_{L^2}\left||\mathbf{u}'|^3\right|_{L^{\frac{10}{3}}}^{\frac{5}{3}} \\
&\le C|\mathbf{u}'|_{L^2}^2\,|\mathbf{u}'|_{L^6}^4 + \frac{\mu}{4}\int_\Omega |\mathbf{u}'|^4|\nabla\mathbf{u}'|^2\,\mathrm{d}\Omega + \frac{\nu}{4}\int_\Omega |\mathbf{u}'|^4\left|\frac{\partial\mathbf{u}'}{\partial z}\right|^2\mathrm{d}\Omega.
\end{aligned}$$

In the above we have used the interpolation inequality

$$|\phi|_{L^{\frac{10}{3}}} \le C|\phi|_{L^2}^{\frac{2}{5}}|\phi|_{H^1}^{\frac{3}{5}}, \tag{2.55}$$

which can be obtained by setting $p = 10/3$, $p_1 = 2$, $p_2 = 6$ in (A.1), and then using (A.5). Similarly,

$$\begin{aligned}
&\int_\Omega -\nu\frac{\partial^2}{\partial z^2}P_{M,N}\mathbf{u}' \cdot |\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega \\
&\qquad \le C|\mathbf{u}'|_{L^2}^2\,|\mathbf{u}'|_{L^6}^4 + \frac{\mu}{4}\int_\Omega |\mathbf{u}'|^4|\nabla\mathbf{u}'|^2\,\mathrm{d}\Omega + \frac{\nu}{4}\int_\Omega |\mathbf{u}'|^4\left|\frac{\partial\mathbf{u}'}{\partial z}\right|^2\mathrm{d}\Omega.
\end{aligned}$$

Hence we derive from (2.48) that

$$\begin{aligned}
&\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}'|_{L^6}^6 + \\
&\quad \int_\Omega \left((\mathbf{u}'\cdot\nabla)\overline{\mathbf{u}} - \overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}' + (\nabla\cdot\mathbf{u}')\mathbf{u}'}\right)\cdot|\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega + \\
&\qquad \frac{1}{2}\mu\int_\Omega\left(|\nabla\mathbf{u}'|^2|\mathbf{u}'|^4 + \left|\nabla|\mathbf{u}'|^2\right|^2|\mathbf{u}'|^2\right)\mathrm{d}\Omega + \\
&\qquad \frac{1}{2}\nu\int_\Omega\left(\left|\frac{\partial\mathbf{u}'}{\partial z}\right|^2|\mathbf{u}'|^4 + \left|\frac{\partial}{\partial z}|\mathbf{u}'|^2\right|^2|\mathbf{u}'|^2\right)\mathrm{d}\Omega + \\
&\qquad\qquad \le \int_\Omega \mathbf{F}'\cdot|\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega + C|\mathbf{u}'|_{L^2}^2|\mathbf{u}'|_{L^6}^4.
\end{aligned} \tag{2.56}$$

For the integrals on the right–hand side of (2.56) that involve nonlinear terms, we proceed by integration by parts, using the periodic boundary

conditions on $\mathbf{u}'$ and $\overline{\mathbf{u}}$ when appropriate, and we find that

$$\begin{aligned}
&\left|\int_\Omega (\mathbf{u}'\cdot\nabla)\overline{\mathbf{u}}\cdot|\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}\Omega\right| \\
&= \left|\int_\Omega \big((\nabla\cdot\mathbf{u}')\overline{\mathbf{u}}\cdot|\mathbf{u}'|^4\mathbf{u}' + (\mathbf{u}'\cdot\nabla)(|\mathbf{u}'|^4\mathbf{u}')\cdot\overline{\mathbf{u}}\big)\,\mathrm{d}\Omega\right| \\
&\le \left|\int_M \left(\overline{\mathbf{u}}\cdot\int_{-H}^0 (\nabla\cdot\mathbf{u}')|\mathbf{u}'|^4\mathbf{u}'\,\mathrm{d}z + \overline{\mathbf{u}}\cdot\int_{-H}^0 (\mathbf{u}'\cdot\nabla)(|\mathbf{u}'|^4\mathbf{u}')\,\mathrm{d}z\right)\mathrm{d}x\,\mathrm{d}y\right| \\
&\le C\int_M |\overline{\mathbf{u}}|\int_{-H}^0 |\nabla\mathbf{u}'||\mathbf{u}'|^5\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

By the Cauchy–Schwarz inequality and the Holder's inequality, we find

$$\begin{aligned}
&\left|\int_M \overline{\mathbf{u}}\int_{-H}^0 |\nabla\mathbf{u}'||\mathbf{u}'|^5\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\right| \le \\
&\left(\int_M |\overline{\mathbf{u}}|^4\,\mathrm{d}M\right)^{\frac14}\left(\int_\Omega |\nabla\mathbf{u}'|^2|\mathbf{u}'|^4\,\mathrm{d}\Omega\right)^{\frac12}\left(\int_M\left(\int_{-H}^0 |\mathbf{u}'|^6\,\mathrm{d}z\right)^2\mathrm{d}M\right)^{\frac14}.
\end{aligned} \tag{2.57}$$

By the Minkowski integral inequality (A.6), we have

$$\left(\int_M\left(\int_{-H}^0 |\mathbf{u}'|^6\,\mathrm{d}z\right)^2\mathrm{d}M\right)^{\frac12} \le \int_{-H}^0\left(\int_M |\mathbf{u}'|^{12}\,\mathrm{d}M\right)^{\frac12}\mathrm{d}z. \tag{2.58}$$

Applying the Ladyzhenskaya inequality (A.2) to ϕ^3 in $\mathbb{R}^2$, we obtain

$$|\phi|^{12}_{L^{12}(M)} \le C|\phi|^6_{L^6(M)}\left(\int_M |\phi|^4|\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}y\right) + |\phi|^{12}_{L^6(M)}. \tag{2.59}$$

Using (2.59), we infer from (2.58) that

$$\begin{aligned}
&\left(\int_M\left(\int_{-H}^0 |\mathbf{u}'|^6\,\mathrm{d}z\right)^2\mathrm{d}M\right)^{\frac12} \\
&\le \int_{-H}^0\left(C\int_M |\mathbf{u}'|^6\,\mathrm{d}M\int_M |\mathbf{u}'|^4|\nabla\mathbf{u}'|^2\,\mathrm{d}M + \left(\int_M |\mathbf{u}'|^6\,\mathrm{d}M\right)^2\right)^{\frac12}\mathrm{d}z \\
&\le C\int_{-H}^0\left(\int_M |\mathbf{u}'|^6\,\mathrm{d}M\right)^{\frac12}\left(\int_M |\mathbf{u}'|^4|\nabla\mathbf{u}'|^2\,\mathrm{d}M\right)^{\frac12} + \int_\Omega |\mathbf{u}'|^6\,\mathrm{d}\Omega \\
&\le C\left(\int_\Omega |\mathbf{u}'|^6\,\mathrm{d}\Omega\right)^{\frac12}\left(\int_\Omega |\mathbf{u}'|^4|\nabla\mathbf{u}'|^2\,\mathrm{d}\Omega\right)^{\frac12} + \int_\Omega |\mathbf{u}'|^6\,\mathrm{d}\Omega.
\end{aligned}$$

Therefore we have

$$\left(\int_M \left(\int_{-H}^0 |\mathbf{u}'|^6 \,\mathrm{d}z\right)^2 \mathrm{d}M\right)^{\frac{1}{2}} \leq \\ C|\mathbf{u}'|_{L^6}^3 \left(\int_\Omega |\mathbf{u}'|^4 |\nabla \mathbf{u}'|^2 \,\mathrm{d}\Omega\right)^{\frac{1}{2}} + |\mathbf{u}'|_{L^6}^6. \quad (2.60)$$

By the Ladyzhenskaya inequality (A.2) for functions in $\mathbb{R}^2$, we find that

$$\left(\int_M |\overline{\mathbf{u}}|^4 \,\mathrm{d}M\right)^{\frac{1}{4}} \leq C|\overline{\mathbf{u}}|_{L^2(M)}^{\frac{1}{2}} |\overline{\mathbf{u}}|_{H^1(M)}^{\frac{1}{2}}, \quad (2.61)$$

Using (2.60) and (2.61) we infer from (2.57) that

$$\int_M \left(\overline{\mathbf{u}} \int_{-H}^0 |\nabla \mathbf{u}'||\mathbf{u}'|^5 \,\mathrm{d}z\right) \mathrm{d}x\,\mathrm{d}y \\ \leq C|\overline{\mathbf{u}}|_{L^2(M)}^{\frac{1}{2}} |\overline{\mathbf{u}}|_{H^1(M)}^{\frac{1}{2}} \left(|\mathbf{u}'|_{L^6}^{\frac{3}{2}} \left(\int_\Omega |\mathbf{u}'|^4 |\nabla \mathbf{u}'|^2 \,\mathrm{d}\Omega\right)^{\frac{3}{4}} + |\mathbf{u}'|_{L^6}^6\right). \quad (2.62)$$

By Young's inequality, we have

$$|\overline{\mathbf{u}}|_{L^2(M)}^{\frac{1}{2}} |\overline{\mathbf{u}}|_{H^1(M)}^{\frac{1}{2}} |\mathbf{u}'|_{L^6}^{\frac{3}{2}} \left(\int_\Omega |\mathbf{u}'|^4 |\nabla \mathbf{u}'|^2 \,\mathrm{d}\Omega\right)^{\frac{3}{4}} \\ \leq C|\overline{\mathbf{u}}|_{L^2(M)}^2 |\overline{\mathbf{u}}|_{H^1(M)}^2 |\mathbf{u}'|_{L^6}^6 + \frac{1}{4}\mu \int_\Omega |\mathbf{u}'|^4 |\nabla \mathbf{u}'|^2 \,\mathrm{d}\Omega, \quad (2.63)$$

For the other integral that involves nonlinear terms,

$$\left|\int_\Omega \overline{(\mathbf{u}' \cdot \nabla)\mathbf{u}' + (\nabla \cdot \mathbf{u}')\mathbf{u}'} \cdot |\mathbf{u}'|^4 \mathbf{u}' \,\mathrm{d}\Omega\right| \\ = \left|\int_\Omega \overline{u_i' u_j'} \partial_i (|\mathbf{u}'|^4 u_j') \,\mathrm{d}\Omega\right| \\ \leq \left|\int_M \overline{u_i' u_j'} \left(\int_{-H}^0 \partial_j (|\mathbf{u}'|^4 u_j') \,\mathrm{d}z\right) \mathrm{d}x\,\mathrm{d}y\right| \\ \leq \left|\frac{1}{H} \int_M \left(\int_{-H}^0 u_i' u_j' \,\mathrm{d}z \int_{-H}^0 \partial_j (|\mathbf{u}'|^4 u_j') \,\mathrm{d}z\right) \mathrm{d}x\,\mathrm{d}y\right| \\ \leq C \int_M \left(\int_{-H}^0 |\mathbf{u}'|^2 \,\mathrm{d}z \int_{-H}^0 |\mathbf{u}'|^4 |\nabla \mathbf{u}'|^2 \,\mathrm{d}z\right) \mathrm{d}x\,\mathrm{d}y.$$

By similar use of the Minkowski inequality (A.6) and various interpolation inequalities, we find that

$$\int_M \left(\int_{-H}^0 |\mathbf{u}'|^2 \, \mathrm{d}z \int_{-H}^0 |\mathbf{u}'|^4 |\nabla \mathbf{u}'|^2 \, \mathrm{d}z \right) \mathrm{d}x \, \mathrm{d}y \leq$$
$$C|\mathbf{u}'|^3_{L^6(\Omega)} \left(\int_\Omega |\mathbf{u}'|^4 |\nabla \mathbf{u}'|^2 \, \mathrm{d}\Omega \right)^{\frac{1}{2}} \left(|\mathbf{u}'|_{L^2(\Omega)} + |\nabla \mathbf{u}'|_{L^2(\Omega)} \right). \quad (2.64)$$

By Young's inequality, we have

$$\left(|\mathbf{u}'|_{L^2(\Omega)} + |\nabla \mathbf{u}'|_{L^2(\Omega)} \right) |\mathbf{u}'|^3_{L^6(\Omega)} \left(\int_\Omega |\mathbf{u}'|^4 |\nabla \mathbf{u}'|^2 \, \mathrm{d}\Omega \right)^{\frac{1}{2}} \leq$$
$$\left(|\mathbf{u}'|^2_{L^2(\Omega)} + |\nabla \mathbf{u}'|^2_{L^2(\Omega)} \right) |\mathbf{u}'|^6_{L^6(\Omega)} + \frac{1}{4}\mu \int_\Omega |\mathbf{u}'|^4 |\nabla \mathbf{u}'|^2 \, \mathrm{d}\Omega. \quad (2.65)$$

Putting (2.56)–(2.65) together, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}'|^6_{L^6} + \mu \int_\Omega \left(|\nabla \mathbf{u}'|^2 |\mathbf{u}'|^4 + \left|\nabla |\mathbf{u}'|^2\right|^2 |\mathbf{u}'|^2 \right) \mathrm{d}\Omega +$$
$$\nu \int_\Omega \left(\left| \frac{\partial \mathbf{u}'}{\partial z} \right|^2 |\mathbf{u}'|^4 + \left| \frac{\partial}{\partial z} |\mathbf{u}'|^2 \right|^2 |\mathbf{u}'|^2 \right) \mathrm{d}\Omega \leq \left(C|\mathbf{F}'|^2_{L^2} + C_2 |\mathbf{u}'|^2_{L^2} \right) |\mathbf{u}'|^4_{L^6} +$$
$$C \left(|\overline{\mathbf{u}}|^{\frac{1}{2}}_{L^2(M)} |\overline{\mathbf{u}}|^{\frac{1}{2}}_{H^1(M)} + |\overline{\mathbf{u}}|^2_{L^2(M)} |\overline{\mathbf{u}}|^2_{H^1(M)} + |\mathbf{u}'|^2_{L^2(\Omega)} + C|\nabla \mathbf{u}'|^2_{L^2(\Omega)} \right) |\mathbf{u}'|^6_{L^6}. \quad (2.66)$$

Ignoring the other positive terms on the left hand side of (2.66), and dividing both sides by $|\mathbf{u}'|^4_{L^6}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}'|^2_{L^6} \leq C|\mathbf{F}'|^2_{L^2} + C_2 |\mathbf{u}'|^2_{L^2} +$$
$$C \left(|\mathbf{u}|^2_{L^2(\Omega)} |\mathbf{u}|^2_{H^1(\Omega)} + |\mathbf{u}|^2_{L^2(\Omega)} + |\nabla \mathbf{u}|^2_{L^2(\Omega)} \right) |\mathbf{u}'|^2_{L^6}. \quad (2.67)$$

Applying the Gronwall inequality to (2.67), and using the L^2 estimate result (2.46), we obtain

$$|\mathbf{u}'(\cdot, t)|^2_{L^6} \leq J_6, \quad (2.68)$$

with

$$J_6(t) = e^{C\left(K_1^2(t) + K_1(t)t + K_1(t)\right)} \left(|\mathbf{u}_0|^2_{L^6(\Omega)} + C_2 J_1(t) + C \int_0^t |\mathbf{F}|^2_{L^2(\Omega)} \, \mathrm{d}s \right).$$

Integrating (2.66) over $[0,t]$, and using the estimate (2.68), we obtain

$$\mu\int_0^t\int_\Omega\left(|\nabla\mathbf{u}'|^2|\mathbf{u}'|^4+\left|\nabla|\mathbf{u}'|^2\right|^2|\mathbf{u}'|^2\right)\mathrm{d}\Omega\,\mathrm{d}s+ \\ \nu\int_0^t\int_\Omega\left(\left|\frac{\partial\mathbf{u}'}{\partial z}\right|^2|\mathbf{u}'|^4+\left|\frac{\partial}{\partial z}|\mathbf{u}'|^2\right|^2|\mathbf{u}'|^2\right)\mathrm{d}\Omega\,\mathrm{d}s\le\widetilde{J}_6(t), \quad (2.69)$$

with

$$\widetilde{J}_6(t)=|\mathbf{u}_0|_{L^6}^6+J_6^2(t)\left(C\int_0^t|F(\cdot,s)|_{L^2}^2\,\mathrm{d}s+C_2K_1t\right)+ \\ CJ_6^3(t)\left(K_1^2(t)+K_1(t)t+K_1(t)\right).$$

Estimate $|\nabla\overline{\mathbf{u}}|_{L^2(M)}$
We multiply (2.26) by $-\Delta\overline{\mathbf{u}}$ and integrate by parts over M to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\overline{\mathbf{u}}|_{L^2(M)}^2+\mu|\Delta\overline{\mathbf{u}}|_{L^2(M)}^2+\int_M f\mathbf{k}\times\overline{\mathbf{u}}\cdot(-\Delta\overline{\mathbf{u}})\,\mathrm{d}M= \\ \int_M\overline{\mathbf{F}}\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M-\frac{1}{\rho_0}\int_M\nabla p_0\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M-\int_M(\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}}\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M \\ -\int_M\overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}'+(\nabla\cdot\mathbf{u}')\mathbf{u}'}\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M+\mu\int_\Omega\Delta P_{M,N}\overline{\mathbf{u}}\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}\Omega. \quad (2.70)$$

We note that, by integration by parts,

$$\left|\int_M(f\mathbf{k}\times\overline{\mathbf{u}})\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}\Omega\right|\le|f|_\infty|\overline{\mathbf{u}}|_{L^2}|\Delta\overline{\mathbf{u}}|_{L^2} \\ \le C|f|_\infty^2|\overline{\mathbf{u}}|_{L^2}^2+\frac{\mu}{4}|\Delta\overline{\mathbf{u}}|_{L^2}^2.$$

And thanks to (2.21),

$$\int_M\nabla p_0\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M=0.$$

Following similar steps in the handling of the 2D Navier Stokes equations, we obtain

$$\int_M(\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}}\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M\le C|\overline{\mathbf{u}}|_{L^2(M)}^{\frac{1}{2}}|\nabla\overline{\mathbf{u}}|_{L^2(M)}|\Delta\overline{\mathbf{u}}|_{L^2(M)}^{\frac{3}{2}}.$$

Applying the Cauchy–Schwarz and Holder inequalities one have

$$\int_M \overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}' + (\nabla\cdot\mathbf{u}')\mathbf{u}'}\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M \le \\ C|\nabla\mathbf{u}'|^{\frac{1}{2}}_{L^2(\Omega)}\left(\int_\Omega |\mathbf{u}'|^4|\nabla\mathbf{u}'|^2\,\mathrm{d}\Omega\right)^{\frac{1}{4}}|\Delta\overline{\mathbf{u}}|_{L^2(M)}.$$

For this last integral in (2.70), we proceed by the Cauchy–Schwarz inequality and (2.44) with $k=2$,

$$\begin{aligned}\mu\int_\Omega \Delta P_{M,N}\overline{\mathbf{u}}\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}\Omega &\le \mu|\Delta P_{M,N}\overline{\mathbf{u}}|_{L^2}|\Delta\overline{\mathbf{u}}|^2_{L^2}\\ &\le C_2\mu|\overline{\mathbf{u}}|_{L^2}|\Delta\overline{\mathbf{u}}|^2_{L^2}\\ &\le C_2\mu|\overline{\mathbf{u}}|^2_{L^2}+\frac{\mu}{4}|\Delta\overline{\mathbf{u}}|^2_{L^2}.\end{aligned}$$

Then we derive from (2.70) that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\overline{\mathbf{u}}|^2_{L^2(M)}+\mu|\Delta\overline{\mathbf{u}}|^2_{L^2(M)} \le \frac{2}{\mu}|\overline{\mathbf{F}}|^2_{L^2(M)}+(C_2\mu+C|f|^2_\infty)|\mathbf{u}|^2_{L^2}+ \\ C|\overline{\mathbf{u}}|^2_{L^2(M)}|\nabla\overline{\mathbf{u}}|^4_{L^2(M)}+C|\nabla\mathbf{u}'|^2_{L^2(\Omega)}+C\int_\Omega|\mathbf{u}'|^4|\nabla\mathbf{u}'|^2\,\mathrm{d}\Omega. \quad (2.71)$$

Applying the Gronwall inequality to (2.71), and using the previous estimates (2.46) and (2.69), we obtain

$$|\nabla\overline{\mathbf{u}}(\cdot,t)|^2_{L^2(M)}+\frac{\mu}{2}\int_0^t|\Delta\overline{\mathbf{u}}|^2_{L^2(M)}\,\mathrm{d}s\le J_2(t), \quad (2.72)$$

with

$$J_2(t)=e^{CJ_1^2(t)}\Big(|\mathbf{u}_0|^2_{H^1}+(C_2\mu+C|f|^2_\infty)J_1(t)t+ \\ C\int_0^t|\mathbf{F}|^2_{L^2(M)}\,\mathrm{d}s+CJ_1(t)+C\widetilde{J}_6(t)\Big).$$

To estimate $|\partial_z\mathbf{u}|_{L^2(\Omega)}$

We multiply (2.14) by $\partial^2\mathbf{u}/\partial z^2$ and integrate by parts over Ω, and we

have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\partial_z\mathbf{u}|^2_{L^2(\Omega)} + \mu|\nabla\partial_z\mathbf{u}|^2_{L^2(\Omega)} + \nu\left|\frac{\partial^2\mathbf{u}}{\partial z^2}\right|^2_{L^2(\Omega)} = \left(F, \frac{\partial^2\mathbf{u}}{\partial z^2}\right) -$$
$$\int_\Omega \left((\mathbf{u}\cdot\nabla)\mathbf{u} + \left(\nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}z\right)\frac{\partial\mathbf{u}}{\partial z}\right)\cdot\frac{\partial^2\mathbf{u}}{\partial z^2}\,\mathrm{d}\Omega +$$
$$\mu|\nabla P_{M,N}\partial_z\mathbf{u}|^2_{L^2(\Omega)} + \nu\left|\frac{\partial^2}{\partial z^2}P_{M,N}\mathbf{u}\right|^2_{L^2(\Omega)}. \quad (2.73)$$

By Holder's inequality,

$$\left(F, \frac{\partial^2\mathbf{u}}{\partial z^2}\right) \le C|F|^2_{L^2} + \frac{\nu}{4}\left|\frac{\partial^2\mathbf{u}}{\partial z^2}\right|^2_{L^2}.$$

With regard to the integral in (2.73) that involves the nonlinear convection terms, we find

$$-\int_\Omega \left((\mathbf{u}\cdot\nabla)\mathbf{u} + \left(\nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}z\right)\frac{\partial\mathbf{u}}{\partial z}\right)\cdot\frac{\partial^2\mathbf{u}}{\partial z^2}\,\mathrm{d}\Omega$$
$$= \int_\Omega \frac{\partial}{\partial z}\left((\mathbf{u}\cdot\nabla)\mathbf{u} + \left(\nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}z\right)\frac{\partial\mathbf{u}}{\partial z}\right)\cdot\frac{\partial\mathbf{u}}{\partial z}\,\mathrm{d}\Omega$$
$$= \int_\Omega \left((\partial_z\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\partial_z\mathbf{u} - (\nabla\cdot\mathbf{u})\frac{\partial\mathbf{u}}{\partial z} + \nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}\xi\frac{\partial^2\mathbf{u}}{\partial z^2}\right)\cdot\frac{\partial\mathbf{u}}{\partial z}\,\mathrm{d}\Omega$$
$$= \int_\Omega \left((\partial_z\mathbf{u}\cdot\nabla)\mathbf{u} - (\nabla\cdot\mathbf{u})\frac{\partial\mathbf{u}}{\partial z}\right)\cdot\frac{\partial\mathbf{u}}{\partial z}\,\mathrm{d}\Omega$$
$$= \int_\Omega -(\nabla\cdot\partial_z\mathbf{u})\mathbf{u}\cdot\partial_z\mathbf{u} - (\partial_z\mathbf{u}\cdot\nabla)\partial_z\mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\nabla)\partial_z\mathbf{u}\cdot\partial_z\mathbf{u}\,\mathrm{d}\Omega.$$

Hence

$$\left|\int_\Omega \left((\mathbf{u}\cdot\nabla)\mathbf{u} + \left(\nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}z\right)\frac{\partial\mathbf{u}}{\partial z}\right)\cdot\frac{\partial^2\mathbf{u}}{\partial z^2}\,\mathrm{d}\Omega\right|$$
$$\le C\int_\Omega |\mathbf{u}||\nabla\partial_z\mathbf{u}||\partial_z\mathbf{u}|\,\mathrm{d}\Omega$$
$$\le C|\mathbf{u}|_{L^6(\Omega)}|\partial_z\mathbf{u}|_{L^3(\Omega)}|\nabla\partial_z\mathbf{u}|_{L^2(\Omega)}$$
$$\le C|\mathbf{u}|_{L^6(\Omega)}|\partial_z\mathbf{u}|^{\frac{1}{2}}_{L^2(\Omega)}|\partial_z\mathbf{u}|^{\frac{1}{2}}_{H^1(\Omega)}|\nabla\partial_z\mathbf{u}|_{L^2(\Omega)}$$
$$\le C|\mathbf{u}|_{L^6(\Omega)}|\partial_z\mathbf{u}|^{\frac{1}{2}}_{L^2(\Omega)}|\mathbf{u}_{zz}|^{\frac{1}{2}}_{L^2}|\nabla\partial_z\mathbf{u}|_{L^2(\Omega)} + C|\mathbf{u}|_{L^6(\Omega)}|\partial_z\mathbf{u}|^{\frac{1}{2}}_{L^2(\Omega)}|\nabla\partial_z\mathbf{u}|^{\frac{3}{2}}_{L^2(\Omega)}$$
$$\le C|\mathbf{u}|^4_{L^6}|\partial_z\mathbf{u}|^2_{L^2} + \frac{\nu}{4}|\mathbf{u}_{zz}|^2_{L^2} + \frac{\mu}{2}|\nabla\partial_z\mathbf{u}|^2_{L^2}.$$

The last two integrals on the right–hand side of (2.73) can be handled by (2.44) with $k=2$. Thus we derive from (2.73) that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\partial_z \mathbf{u}|^2_{L^2(\Omega)} + \mu|\nabla\partial_z\mathbf{u}|^2_{L^2(\Omega)} + \nu\left|\frac{\partial^2\mathbf{u}}{\partial z^2}\right|^2_{L^2(\Omega)} \le C|\mathbf{F}|^2_{L^2} + \\ C\left(J_2^2(t)+J_6^2(t)\right)|\partial_z\mathbf{u}|^2_{L^2} + C_2|\mathbf{u}|^2_{L^2}. \quad (2.74)$$

An application of the Gronwall inequality to (2.74) readily yields

$$|\partial_z\mathbf{u}(\cdot,t)|^2_{L^2(\Omega)} + \mu\int_0^t|\nabla\partial_z\mathbf{u}|^2_{L^2(\Omega)}\,\mathrm{d}s + \nu\int_0^t|\frac{\partial^2\mathbf{u}}{\partial z^2}|^2_{L^2(\Omega)}\,\mathrm{d}s \le J_z(t), \quad (2.75)$$

with

$$J_z(t) = e^{C\left(J_2^2(t)+J_6^2(t)\right)t}\left(|\mathbf{u}_0|^2_{H^1} + C\int_0^t|\mathbf{F}|^2_{L^2}\,\mathrm{d} + C_2J_1t\right).$$

To estimate $|\nabla\mathbf{u}|_{L^2}$
We multiply (2.14) by $-\Delta\mathbf{u}$ and integrate by parts over Ω,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{u}|^2_{L^2(\Omega)} + \int_\Omega\left((\mathbf{u}\cdot\nabla)\mathbf{u} + \left(\nabla\cdot\int_z^0\mathbf{u}\,\mathrm{d}\xi\right)\frac{\partial\mathbf{u}}{\partial z}\right)\cdot(-\Delta\mathbf{u})\,\mathrm{d}\Omega + \\ f\int_\Omega k\times\mathbf{u}\cdot(-\Delta\mathbf{u})\,\mathrm{d}\Omega + \frac{1}{\rho_0}\int_\Omega\nabla p_0\cdot(-\Delta\mathbf{u})\,\mathrm{d}\Omega + \mu|\Delta\mathbf{u}|^2_{L^2(\Omega)} + \nu|\nabla\partial_z\mathbf{u}|^2_{L^2(\Omega)} \\ = (F,-\Delta\mathbf{u}) + \mu|\Delta P_{M,N}\mathbf{u}|^2_{L^2(\Omega)} + \nu|\nabla P_{M,N}\partial_z\mathbf{u}|^2_{L^2(\Omega)}. \quad (2.76)$$

We note that

$$\int_M (f\mathbf{k}\times\mathbf{u})\cdot\Delta\mathbf{u}\,\mathrm{d}\Omega = 0,$$

and

$$\int_\Omega\nabla p_0\cdot(-\Delta\mathbf{u})\,\mathrm{d}\Omega = -\int_M\nabla p_0\cdot\int_{-H}^0(\Delta\mathbf{u})\,\mathrm{d}z\,\mathrm{d}M \\ = -H\int_M\nabla p_0\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M \\ = 0.$$

It is easy to see that

$$\int_\Omega\left((\mathbf{u}\cdot\nabla)\mathbf{u} + \left(\nabla\cdot\int_z^0\mathbf{u}\,\mathrm{d}\xi\right)\frac{\partial\mathbf{u}}{\partial z}\right)\cdot(\Delta\mathbf{u})\,\mathrm{d}\Omega \le \\ C\int_\Omega\left(|\mathbf{u}||\nabla\mathbf{u}| + \int_{-H}^0|\nabla\mathbf{u}|\,\mathrm{d}z|\partial_z\mathbf{u}|\right)|\Delta\mathbf{u}|\,\mathrm{d}\Omega.$$

By the Young's inequality and the Sobolev interpolation inequality (A.4) for L^3 functions, we have

$$\int_\Omega |\mathbf{u}||\nabla\mathbf{u}||\Delta\mathbf{u}|\,\mathrm{d}\Omega \le C|\mathbf{u}|_{L^6}|\nabla\mathbf{u}|_{L^3}|\Delta\mathbf{u}|_{L^2}$$
$$\le C|\mathbf{u}|_{L^6}|\nabla\mathbf{u}|_{L^2}^{\frac{1}{2}}|\nabla\partial_z\mathbf{u}|_{L^2}^{\frac{1}{2}}|\Delta\mathbf{u}|_{L^2} + C|\mathbf{u}|_{L^6}|\nabla\mathbf{u}|_{L^2}^{\frac{1}{2}}|\Delta\mathbf{u}|_{L^2}^{\frac{3}{2}}.$$

We appeal to the following inequality for functions in $\mathbb{R}^3$ (the scaling is 2D due to the vertical integration. For a proof, see [6]),

$$\int_\Omega\left(\int_{-H}^0 |\nabla\mathbf{u}|\,\mathrm{d}z\right)|f||g|\,\mathrm{d}\Omega \le C|f|_{L^2}|\mathbf{u}|_{H^1}^{\frac{1}{2}}|\mathbf{u}|_{H^2}^{\frac{1}{2}}|g|_{L^2}^{\frac{1}{2}}|g|_{H^1}^{\frac{1}{2}},$$

to obtain (by setting $f=\Delta\mathbf{u}$, $g=\partial_z\mathbf{u}$)

$$\int_\Omega\left(\int_{-H}^0 |\nabla\mathbf{u}|\,\mathrm{d}z\right)|\partial_z\mathbf{u}||\Delta\mathbf{u}|\,\mathrm{d}\Omega \le C|\nabla\mathbf{u}|_{L^2}^{\frac{1}{2}}|\partial_z\mathbf{u}|_{L^2}^{\frac{1}{2}}|\nabla\partial_z\mathbf{u}|_{L^2}^{\frac{1}{2}}|\Delta\mathbf{u}|_{L^2}^{\frac{3}{2}}.$$

The last two integrals in the equation above are handled by (2.44) with $k=2$. After these intermediate steps we derive from (2.76) that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{u}|_{L^2}^2 + \mu|\Delta\mathbf{u}|_{L^2}^2 + \nu|\nabla\partial_z\mathbf{u}|_{L^2}^2 \le \frac{4}{\mu}|\mathbf{F}|_{L^2}^2 +$$
$$C\left(|\mathbf{u}|_{L^6}^4 + |\partial_z\mathbf{u}|_{L^2}^2|\nabla\partial_z\mathbf{u}|_{L^2}^2\right)|\nabla\mathbf{u}|_{L^2}^2 + C_2|\mathbf{u}|_{L^2}^2. \quad (2.77)$$

We appeal to the Gronwall inequality, and, with use of (2.46), (2.68), and (2.75), we obtain

$$|\nabla\mathbf{u}(\cdot,t)|_{L^2}^2 + \mu\int_0^t |\Delta\mathbf{u}|_{L^2}^2\,\mathrm{d}s + \nu\int_0^t |\nabla\partial_z\mathbf{u}|_{L^2}^2\,\mathrm{d}s \le J_V(t), \quad (2.78)$$

with

$$J_V(t) = e^{C\left(J_6^2+J_2^2(t)\right)t+J_z^2(t)}\left(|\mathbf{u}_0|_{H^1}^2 + C_2J_1t + \frac{4}{\mu}\int_0^t |\mathbf{F}(\cdot,s)|_{L^2}^2\,\mathrm{d}s\right).$$

We now prove Theorem 2.1.

Proof. The short time existence and uniqueness of the strong solutions of (2.1)–(2.8) can be established as for the viscous primitive equations (see [14, 17]). Let $\mathbf{u}$ be such a strong solution corresponding to the initial data $\mathbf{u}_0$ with the maximal interval of existence $[0,T^*)$. If $T^*\ge T$, then there is nothing to prove here. Let us suppose that $T^*<T$, and in particular, $T^*<\infty$. Then it is clear that

$$\limsup_{t\to T^*-} ||\mathbf{u}||_{H^1} = \infty. \quad (2.79)$$

Otherwise the solution can be extended beyond T^*. However the estimates (2.75) and (2.78) indicate that $||\mathbf{u}(\cdot,t)||_{H^1}<\infty$ for all $t<T$,

which contradicts (2.79). Hence the solution must exist for the whole period of $[0, T)$.

It remains to show the continuous dependence of the solution on the data, of which the uniqueness of the solution is a consequence. Let us assume that $\mathbf{u}^1$ and $\mathbf{u}^2$ be two solutions corresponding to the two sets of initial data $\mathbf{u}_0^1$ and $\mathbf{u}_0^2$, respectively. We let

$$\mathbf{u} = \mathbf{u}^1 - \mathbf{u}^2.$$

Then $\mathbf{u}$ satisfies the following equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}^1 \cdot \nabla)\mathbf{u} + (\nabla \cdot \int_z^0 \mathbf{u}^1 \, \mathrm{d}\xi)\frac{\partial \mathbf{u}}{\partial z} + (\mathbf{u} \cdot \nabla)\mathbf{u}^2 + (\nabla \cdot \int_z^0 \mathbf{u} \, \mathrm{d}\xi)\frac{\partial \mathbf{u}^2}{\partial z}$$
$$+ f\mathbf{k} \times \mathbf{u} - \mu\Delta\mathbf{u} - \nu\frac{\partial^2 \mathbf{u}}{\partial z^2} = -\mu\Delta P_{M,N}\mathbf{u} - \nu\frac{\partial^2}{\partial z^2}P_{M,N}\mathbf{u}. \quad (2.80)$$

We multiply (2.80) by $\mathbf{u}$ and integrate by parts over Ω to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|_{L^2}^2 + \big((\mathbf{u}^1 \cdot \nabla)\mathbf{u}, \mathbf{u}\big) +$$
$$\left((\nabla \cdot \int_z^0 \mathbf{u}^1 \, \mathrm{d}\xi)\frac{\partial \mathbf{u}}{\partial z}, \, \mathbf{u}\right) + \big((\mathbf{u} \cdot \nabla)\mathbf{u}^2, \mathbf{u}\big) + \left((\nabla \cdot \int_z^0 \mathbf{u} \, \mathrm{d}\xi)\frac{\partial \mathbf{u}^2}{\partial z}, \, \mathbf{u}\right) +$$
$$\mu|\nabla \mathbf{u}|^2 + \nu\left|\frac{\partial \mathbf{u}}{\partial z}\right|^2 = \mu|\nabla P_{M,N}\mathbf{u}|^2 + \nu\left|\frac{\partial}{\partial z}P_{M,N}\mathbf{u}\right|^2. \quad (2.81)$$

We verify that

$$\big((\mathbf{u}^1 \cdot \nabla)\mathbf{u}, \mathbf{u}\big) + \left((\nabla \cdot \int_z^0 \mathbf{u}^1 \, \mathrm{d}\xi)\frac{\partial \mathbf{u}}{\partial z}, \, \mathbf{u}\right) = 0, \quad (2.82)$$

$$\big|\big((\mathbf{u} \cdot \nabla)\mathbf{u}^2, \mathbf{u}\big)\big| \le |\nabla \mathbf{u}^2|_{L^2}|\mathbf{u}|_{L^2}^{\frac{1}{2}}|\nabla \mathbf{u}|_{L^2}^{\frac{3}{2}}, \quad (2.83)$$

and that

$$\left((\nabla \cdot \int_z^0 \mathbf{u} \, \mathrm{d}\xi)\frac{\partial \mathbf{u}^2}{\partial z}, \, \mathbf{u}\right) \le C|\mathbf{u}|_{L^2}^{\frac{1}{2}}|\nabla \mathbf{u}|_{L^2}^{\frac{3}{2}}|\partial_z \mathbf{u}^2|_{L^2}^{\frac{1}{2}}|\nabla\partial_z \mathbf{u}^2|_{L^2}^{\frac{1}{2}}. \quad (2.84)$$

Again, the last two integrals on the right–hand side of (2.81) are handled by (2.44) with $k = 1$. Thus we derive from (2.81) that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|_{L^2}^2 + \mu|\nabla \mathbf{u}|^2 + \nu\left|\frac{\partial \mathbf{u}}{\partial z}\right|^2 \le$$
$$C\left(|\nabla \mathbf{u}^2|_{L^2}^4 + |\partial_z \mathbf{u}^2|_{L^2}^2|\nabla\partial_z \mathbf{u}^2|_{L^2}^2\right)|\mathbf{u}|_{L^2}^2 + C_1|\mathbf{u}|_{L^2}^2. \quad (2.85)$$

Thanks to the a priori estimates (2.75) and (2.78) and the Gronwall inequality, we have

$$|\mathbf{u}(\cdot, t)|_{L^2}^2 \le e^{C(K_V^2(t)t + K_z^2(t)) + C_1}|\mathbf{u}(\cdot, 0)|_{L^2}^2. \quad (2.86)$$

This shows that the solution depends on the initial data continuously. When $\mathbf{u}(\cdot,0) = \mathbf{u}^1(\cdot,0) - \mathbf{u}^2(\cdot,0) = 0$,

$$\mathbf{u}(\cdot,t) = 0, \qquad \text{for all } t > 0. \tag{2.87}$$

This shows the uniqueness of the solution. □

3. A spectral-viscosity model for geophysical turbulence

In this section we study a model that has applications in the simulation of geophysical turbulent flows.

3.1. **The model.** The 3D primitive equations with linear spectral eddy viscosity read

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + w\frac{\partial \mathbf{u}}{\partial z} + f\mathbf{k}\times\mathbf{u} + \frac{1}{\rho_0}\nabla p - \mu\Delta\mathbf{u} - \nu\frac{\partial^2\mathbf{u}}{\partial z^2} - L\mathbf{u} = \mathbf{F}, \tag{3.1}$$

$$\frac{\partial p}{\partial z} = -\rho_0 g, \tag{3.2}$$

$$\nabla\cdot\mathbf{u} + \frac{\partial w}{\partial z} = 0, \tag{3.3}$$

with

$$L\mathbf{u} = \mu_\delta\Delta(I - P_{M,N})\mathbf{u} + \nu_\delta\frac{\partial^2}{\partial z^2}(I - P_{M,N})\mathbf{u}.$$

The other notations being the same as those in Section 2, the newly introduced ones μ_δ and ν_δ are the eddy closure parameters in the horizontal and vertical directions, respectively. The subscript δ indicates that the parameters depend on the grid resolution.

As for the model with partial viscosity in Section 2, we consider a rectangular domain $\Omega = M \times (-H, 0)$, with $M = (0, L_x) \times (0, L_y)$, and we also consider periodic boundary conditions in both the x and y directions, and free–slip, non–penetration boundary conditions in the vertical directions. More precisely,

$$\mathbf{u}(x,y,z,t) = \mathbf{u}(x+L_x,y,z,t), \tag{3.4}$$

$$\mathbf{u}(x,y,z,t) = \mathbf{u}(x,y+L_y,z,t), \tag{3.5}$$

$$\frac{\partial\mathbf{u}}{\partial n}|_{z=0} = \frac{\partial\mathbf{u}}{\partial n}|_{z=-H} = 0, \tag{3.6}$$

$$w|_{z=0} = w|_{z=-H} = 0, \tag{3.7}$$

$$p|_{z=0} = p_0. \tag{3.8}$$

As a consequence of the boundary conditions taken here, the spectral low–pass filter $P_{M,N}$ can be, and is defined as in (2.9).

For the model (3.1)–(3.8) we obtain two results. The first is the global well–posedness of the model, and the second is concerned with the convergence of the solutions of (3.1)–(3.8) to those of the viscous primitive equations as μ_δ and ν_δ tend to zero. The first result shall come as no surprise, and therefore its proof will only be briefly sketched. The second result will be discussed in more details.

As usual, p and w can be expressed in terms of $\mathbf{u}$. Specifically, integrating (3.2) from z to 0, and using the boundary condition (3.8) we obtain

$$p(x,y,z,t) = p_0(x,y,t) - \rho_0 g z, \tag{3.9}$$

Integrating (3.3) from z to 0, and using the boundary condition (3.7), we obtain

$$w(x,y,z,t) = \int_z^0 \nabla\cdot\mathbf{u}\,\mathrm{d}\xi = \nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}\xi. \tag{3.10}$$

Setting $z=-H$ in (3.10), and using (3.7) again, we find

$$\nabla\cdot\int_{-H}^0 \mathbf{u}\,\mathrm{d}z = 0. \tag{3.11}$$

We substitute (3.9) and (3.10) into (3.1), and obtain a single closed equation for $\mathbf{u}$:

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + (\nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}\xi)\frac{\partial\mathbf{u}}{\partial z} + f\mathbf{k}\times\mathbf{u} + \frac{1}{\rho_0}\nabla p_0 - \mu\Delta\mathbf{u} - \nu\frac{\partial^2\mathbf{u}}{\partial z^2} - L\mathbf{u} = \mathbf{F}. \tag{3.12}$$

The prognostic variable $\mathbf{u}$ satisfies the equation (3.1), the constraint (3.11), and the boundary conditions (3.4)–(3.6). To complete the system, we also require $\mathbf{u}$ to satisfy the following initial condition:

$$\mathbf{u}(x,y,z,0) = \mathbf{u}_0(x,y,z). \tag{3.13}$$

The diagnostic variables w and p are given by (3.10) and (3.9) respectively.

We now specify the barotropic and baroclinic modes of (3.12). We let

$$\overline{\mathbf{u}}(x,y,t) = \frac{1}{H}\int_{-H}^0 \mathbf{u}(x,y,z,t)\,\mathrm{d}z, \tag{3.14}$$

which denotes the barotropic mode of the primitive variables. We also denote by

$$\mathbf{u}'(x,y,z,t) = \mathbf{u}(x,y,z,t) - \overline{\mathbf{u}}(x,y,t) \tag{3.15}$$

the baroclinic mode of the primitive variables.

As in Section 2, we derive the equations and conditions that the barotropic mode $\overline{\mathbf{u}}$ and the baroclinic mode $\mathbf{u}'$ must satisfy. They are summarized as follows. The equation and boundary conditions for $\overline{\mathbf{u}}$ are

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}} + [\overline{(\mathbf{u}' \cdot \nabla)\mathbf{u}'} + \overline{(\nabla \cdot \mathbf{u}')\mathbf{u}'}] + f\mathbf{k} \times \overline{\mathbf{u}} + \frac{1}{\rho_0}\nabla p_0 - \mu\Delta\overline{\mathbf{u}} = \overline{\mathbf{F}}, \tag{3.16}$$

$$\overline{\mathbf{u}}(x, y, t) = \overline{\mathbf{u}}(x + L_x, y, t), \tag{3.17}$$

$$\overline{\mathbf{u}}(x, y, t) = \overline{\mathbf{u}}(x, y + L_y, t). \tag{3.18}$$

$$\nabla \cdot \overline{\mathbf{u}} = 0. \tag{3.19}$$

The equation, boundary conditions, and constraint for $\mathbf{u}'$ are

$$\begin{aligned}\frac{\partial \mathbf{u}'}{\partial t} &+ (\mathbf{u}' \cdot \nabla)\mathbf{u}' + (\nabla \cdot \int_z^0 \mathbf{u}' \,\mathrm{d}\xi)\frac{\partial \mathbf{u}'}{\partial z} + f\mathbf{k} \times \mathbf{u}' - \mu\Delta\mathbf{u}' - \nu\frac{\partial^2 \mathbf{u}'}{\partial z^2} - L\mathbf{u}' \\ &+ \Big[(\mathbf{u}' \cdot \nabla)\overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}' - \Big(\overline{(\mathbf{u}' \cdot \nabla)\mathbf{u}'} + \overline{(\nabla \cdot \mathbf{u}')\mathbf{u}'}\Big)\Big] = \mathbf{F}',\end{aligned} \tag{3.20}$$

$$\mathbf{u}'(x, y, t) = \mathbf{u}'(x + L_x, y, t), \tag{3.21}$$

$$\mathbf{u}'(x, y, t) = \mathbf{u}'(x, y + L_y, t), \tag{3.22}$$

$$\left.\frac{\partial \mathbf{u}'}{\partial z}\right|_{z=0} = \left.\frac{\partial \mathbf{u}'}{\partial z}\right|_{z=-H} = 0, \tag{3.23}$$

$$\int_{-H}^0 \mathbf{u}' \,\mathrm{d}z = 0. \tag{3.24}$$

3.2. **Existence and uniqueness of strong solutions.** The functional settings are the same as specified in Section 2. A priori estimates are the essential ingredients in the proof of global well–posedness of (3.1)–(3.8). We shall first derive the a priori estimates for $\mathbf{u}$, $\overline{\mathbf{u}}$ and $\mathbf{u}'$.

L^2 estimates

We take inner product of (3.12) with $\mathbf{u}$, integrate by parts over Ω,

using the boundary conditions (3.4)–(3.8), we obtain

$$(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{u}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{u}) + \left((\nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}\xi)\frac{\partial \mathbf{u}}{\partial z},\ \mathbf{u}\right) + \\ (f\mathbf{k}\times\mathbf{u}, \mathbf{u}) + \frac{1}{\rho_0}(\nabla p_0, \mathbf{u}) + \mu|\nabla\mathbf{u}|^2 + \nu\left|\frac{\partial \mathbf{u}}{\partial z}\right|^2 + \\ \mu_\delta|\nabla(I-P_{M,N})\mathbf{u}|^2 + \nu_\delta\left|\frac{\partial}{\partial z}(I-P_{M,N})\mathbf{u}\right|^2 = (\mathbf{F}, \mathbf{u}). \quad (3.25)$$

Following similar steps in Section 2, we obtain that

$$|\mathbf{u}(\cdot,t)|^2+\mu\int_0^t|\nabla\mathbf{u}|^2\,\mathrm{d}s+\nu\int_0^t\left|\frac{\partial}{\partial z}\mathbf{u}\right|^2\,\mathrm{d}s+\mu_\delta\int_0^t|\nabla(I-P_{M,N})\mathbf{u}|^2\,\mathrm{d}s+ \\ \nu_\delta\int_0^t\left|\frac{\partial}{\partial z}(I-P_{M,N})\mathbf{u}\right|^2\,\mathrm{d}s \le K_1(t), \quad (3.26)$$

where

$$K_1(t) \equiv |u_0|^2 + C\int_0^t|\mathbf{F}|^2\,\mathrm{d}s. \quad (3.27)$$

We note that $K_1(t)$ is a non–decreasing positive function in t. This notion will be useful later in the paper.

L^6 estimate on $\mathbf{u}'$
We rewrite (3.20) as

$$\frac{\partial \mathbf{u}'}{\partial t}+(\mathbf{u}'\cdot\nabla)\mathbf{u}'+\left(\nabla\cdot\int_z^0\mathbf{u}'\,\mathrm{d}\xi\right)\frac{\partial \mathbf{u}'}{\partial z}+f\mathbf{k}\times\mathbf{u}'-(\mu+\mu_\delta)\Delta\mathbf{u}'-(\nu+\nu_\delta)\frac{\partial^2\mathbf{u}'}{\partial z^2} \\ +\left[(\mathbf{u}'\cdot\nabla)\overline{\mathbf{u}}+(\overline{\mathbf{u}}\cdot\nabla)\mathbf{u}'-\left(\overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}'}+\overline{(\nabla\cdot\mathbf{u}')\mathbf{u}'}\right)\right] = \mathbf{F}'- \\ \mu_\delta\Delta P_{M,N}\mathbf{u}-\nu_\delta\frac{\partial^2}{\partial z^2}P_{M,N}\mathbf{u}. \quad (3.28)$$

Now (3.28) is in the same form as equation (2.30), and therefore the same techniques from Section (2) can be applied to yield the L^6 estimates on $\mathbf{u}'$. We omit the details and state the results as follows.

$$|\mathbf{u}'(\cdot,t)|_{L^6}^2 \le K_6, \quad (3.29)$$

with

$$K_6(t) = e^{CK_1^2(t)+CK_1(t)t+CK_1(t)+C_1(t)}\left(|\mathbf{u}_0|_{L^6(\Omega)}^2 + C\int_0^t|\mathbf{F}|_{L^2(\Omega)}^2\,\mathrm{d}s\right).$$

We also have

$$(\mu+\mu_\delta)\int_0^t\int_\Omega\left(|\nabla\mathbf{u}'|^2|\mathbf{u}'|^4+\left|\nabla|\mathbf{u}'|^2\right|^2|\mathbf{u}'|^2\right)\,\mathrm{d}\Omega\,\mathrm{d}s+$$
$$(\nu+\nu_\delta)\int_0^t\int_\Omega\left(\left|\frac{\partial\mathbf{u}'}{\partial z}\right|^2|\mathbf{u}'|^4+\left|\frac{\partial}{\partial z}|\mathbf{u}'|^2\right|^2|\mathbf{u}'|^2\right)\,\mathrm{d}\Omega\,\mathrm{d}s\le\widetilde{K}_6(t),\quad(3.30)$$

with

$$\widetilde{K}_6(t)=|\mathbf{u}_0|^6_{L^6}+K_6^2(t)\int_0^t|F(\cdot,s)|^2_{L^2}\,\mathrm{d}s+ K_6^3(t)\left(CK_1^2(t)+CK_1(t)t+CK_1(t)+C_1(t)\right).$$

Estimate $|\nabla\overline{\mathbf{u}}|_{L^2(M)}$
We multiply (3.16) by $-\Delta\overline{\mathbf{u}}$ and integrate by parts over M to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\overline{\mathbf{u}}|^2_{L^2(M)}+\mu|\Delta\overline{\mathbf{u}}|^2_{L^2(M)}-\int_M f\mathbf{k}\times\overline{\mathbf{u}}\cdot\Delta\mathbf{u}\,\mathrm{d}M=$$
$$\int_M\overline{\mathbf{F}}\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M-\frac{1}{\rho_0}\int_M\nabla p_0\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M-\int_M(\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}}\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M-$$
$$\int_M\overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}'+(\nabla\cdot\mathbf{u}')\mathbf{u}'}\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M.\quad(3.31)$$

We note that

$$\int_M f\mathbf{k}\times\overline{\mathbf{u}}\cdot\Delta\mathbf{u}\,\mathrm{d}M=0,$$

and

$$\int_M\nabla p_0\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M=0.$$

Following similar steps in the handling of the 2D Navier Stokes equations, we obtain

$$\int_M(\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}}\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M\le C|\overline{\mathbf{u}}|^{\frac{1}{2}}_{L^2(M)}|\nabla\overline{\mathbf{u}}|_{L^2(M)}|\Delta\overline{\mathbf{u}}|^{\frac{3}{2}}_{L^2(M)}.$$

Applying the Cauchy–Schwarz and Holder inequalities one have

$$\int_M\overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}'+(\nabla\cdot\mathbf{u}')\mathbf{u}'}\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M\le$$
$$C|\nabla\mathbf{u}'|^{\frac{1}{2}}_{L^2(\Omega)}\left(\int_\Omega|\mathbf{u}'|^4|\nabla\mathbf{u}'|^2\,\mathrm{d}\Omega\right)^{\frac{1}{4}}|\Delta\overline{\mathbf{u}}|_{L^2(M)}.$$

Utilizing these estimates, and using Youngs inequality again, we derive from (3.31) that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\overline{\mathbf{u}}|^2_{L^2(M)} + \frac{\mu}{2}|\Delta\overline{\mathbf{u}}|^2_{L^2(M)} \le \frac{2}{\mu}|\overline{\mathbf{F}}|^2_{L^2(M)} + C|\overline{\mathbf{u}}|^2_{L^2(M)}|\nabla\overline{\mathbf{u}}|^4_{L^2(M)} + C|\nabla\mathbf{u}'|^2_{L^2(\Omega)} + C\int_\Omega |\mathbf{u}'|^4|\nabla\mathbf{u}'|^2\,\mathrm{d}\Omega. \quad (3.32)$$

Applying the Gronwall inequality to (3.32), and using the previous estimates (3.26) and (3.30), we obtain

$$|\nabla\overline{\mathbf{u}}(\cdot,t)|^2_{L^2(M)} + \frac{\mu}{2}\int_0^t |\Delta\overline{\mathbf{u}}|^2_{L^2(M)}\,\mathrm{d}s \le K_2(t), \quad (3.33)$$

with

$$K_2(t) = e^{CK_1^2(t)}\left(|\mathbf{u}_0|^2_{H^1} + C\int_0^t |\mathbf{F}|^2_{L^2(M)}\,\mathrm{d}s + CK_1(t) + C\widetilde{K}_6(t)\right).$$

To estimate $|\partial_z\mathbf{u}|_{L^2(\Omega)}$

We multiply (3.12) by $\partial^2\mathbf{u}/\partial z^2$ and integrate by parts over Ω, and utilizing the boundary conditions (3.4)–(3.6), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\partial_z\mathbf{u}|^2_{L^2(\Omega)} + \mu|\nabla\partial_z\mathbf{u}|^2_{L^2(\Omega)} + \nu\left|\frac{\partial^2\mathbf{u}}{\partial z^2}\right|^2_{L^2(\Omega)} + \mu_\delta|\nabla\partial_z(I-P_{M,N})\mathbf{u}|^2_{L^2(\Omega)} + \nu_\delta\left|\frac{\partial^2(I-P_{M,N})\mathbf{u}}{\partial z^2}\right|^2_{L^2(\Omega)} = \left(F, \frac{\partial^2\mathbf{u}}{\partial z^2}\right) - \int_\Omega\left((\mathbf{u}\cdot\nabla)\mathbf{u} + \left(\nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}z\right)\frac{\partial\mathbf{u}}{\partial z}\right)\cdot\frac{\partial^2\mathbf{u}}{\partial z^2}\,\mathrm{d}\Omega. \quad (3.34)$$

By Holder's inequality,

$$\left(F, \frac{\partial^2\mathbf{u}}{\partial z^2}\right) \le C|F|^2_{L^2} + \frac{\nu}{4}\left|\frac{\partial^2\mathbf{u}}{\partial z^2}\right|^2_{L^2}.$$

With regard to the last integral in (3.34) we find

$$
\begin{aligned}
&-\int_\Omega \left((\mathbf{u}\cdot\nabla)\mathbf{u} + \left(\nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}z\right)\frac{\partial\mathbf{u}}{\partial z}\right)\cdot\frac{\partial^2\mathbf{u}}{\partial z^2}\,\mathrm{d}\Omega \\
=&\int_\Omega \frac{\partial}{\partial z}\left((\mathbf{u}\cdot\nabla)\mathbf{u} + \left(\nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}z\right)\frac{\partial\mathbf{u}}{\partial z}\right)\cdot\frac{\partial\mathbf{u}}{\partial z}\,\mathrm{d}\Omega \\
=&\int_\Omega \left((\partial_z\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\partial_z\mathbf{u} - (\nabla\cdot\mathbf{u})\frac{\partial\mathbf{u}}{\partial z} + \nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}\xi\frac{\partial^2\mathbf{u}}{\partial z^2}\right)\cdot\frac{\partial\mathbf{u}}{\partial z}\,\mathrm{d}\Omega \\
=&\int_\Omega \left((\partial_z\mathbf{u}\cdot\nabla)\mathbf{u} - (\nabla\cdot\mathbf{u})\frac{\partial\mathbf{u}}{\partial z}\right)\cdot\frac{\partial\mathbf{u}}{\partial z}\,\mathrm{d}\Omega \\
=&\int_\Omega -(\nabla\cdot\partial_z\mathbf{u})\mathbf{u}\cdot\partial_z\mathbf{u} - (\partial_z\mathbf{u}\cdot\nabla)\partial_z\mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\nabla)\partial_z\mathbf{u}\cdot\partial_z\mathbf{u}\,\mathrm{d}\Omega.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left|\int_\Omega \left((\mathbf{u}\cdot\nabla)\mathbf{u} + \left(\nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}z\right)\frac{\partial\mathbf{u}}{\partial z}\right)\cdot\frac{\partial^2\mathbf{u}}{\partial z^2}\,\mathrm{d}\Omega\right| \\
&\le C\int_\Omega |\mathbf{u}||\nabla\partial_z\mathbf{u}||\partial_z\mathbf{u}|\,\mathrm{d}\Omega \\
&\le C|\mathbf{u}|_{L^6(\Omega)}|\partial_z\mathbf{u}|_{L^3(\Omega)}|\nabla\partial_z\mathbf{u}|_{L^2(\Omega)} \\
&\le C|\mathbf{u}|_{L^6(\Omega)}|\partial_z\mathbf{u}|^{\frac{1}{2}}_{L^2(\Omega)}|\partial_z\mathbf{u}|^{\frac{1}{2}}_{H^1(\Omega)}|\nabla\partial_z\mathbf{u}|_{L^2(\Omega)} \\
&\le C|\mathbf{u}|_{L^6(\Omega)}|\partial_z\mathbf{u}|^{\frac{1}{2}}_{L^2(\Omega)}|\mathbf{u}_{zz}|^{\frac{1}{2}}_{L^2}|\nabla\partial_z\mathbf{u}|_{L^2(\Omega)} + C|\mathbf{u}|_{L^6(\Omega)}|\partial_z\mathbf{u}|^{\frac{1}{2}}_{L^2(\Omega)}|\nabla\partial_z\mathbf{u}|^{\frac{3}{2}}_{L^2(\Omega)} \\
&\le C|\mathbf{u}|^4_{L^6}|\partial_z\mathbf{u}|^2_{L^2} + \frac{\nu}{4}|\mathbf{u}_{zz}|^2_{L^2} + \frac{\mu}{2}|\nabla\partial_z\mathbf{u}|^2_{L^2}.
\end{aligned}
$$

After these intermediate steps, we derive from (3.34) that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\partial_z\mathbf{u}|^2_{L^2(\Omega)} + \mu|\nabla\partial_z\mathbf{u}|^2_{L^2(\Omega)} + \nu\left|\frac{\partial^2\mathbf{u}}{\partial z^2}\right|^2_{L^2(\Omega)} + 2\mu_\delta|\nabla\partial_z(I-P_{M,N})\mathbf{u}|^2_{L^2(\Omega)} + \\
2\nu_\delta\left|\frac{\partial^2(I-P_{M,N})\mathbf{u}}{\partial z^2}\right|^2_{L^2(\Omega)} \le C|F|^2_{L^2} + C|\mathbf{u}|^4_{L^6}|\partial_z\mathbf{u}|^2_{L^2}. \quad (3.35)
\end{aligned}
$$

We note that

$$
\begin{aligned}
|\mathbf{u}|_{L^6} &= |\overline{\mathbf{u}} + \mathbf{u}'|_{L^6} \\
&\le |\overline{\mathbf{u}}|_{L^6} + |\mathbf{u}'|_{L^6} \\
&\le C|\nabla\overline{\mathbf{u}}|_{L^2} + |\mathbf{u}'|_{L^6}.
\end{aligned}
$$

Therefore, by (3.29) and (3.33), we have

$$
|\mathbf{u}|^4_{L^6} \le C\left(K_2^2(t) + K_6^2(t)\right). \quad (3.36)
$$

Then (3.35), together with (3.36), gives

$$\frac{\mathrm{d}}{\mathrm{d}t}|\partial_z \mathbf{u}|^2_{L^2(\Omega)}+\mu|\nabla\partial_z\mathbf{u}|^2_{L^2(\Omega)}+\nu\left|\frac{\partial^2\mathbf{u}}{\partial z^2}\right|^2_{L^2(\Omega)}+2\mu_\delta|\nabla\partial_z(I-P_{M,N})\mathbf{u}|^2_{L^2(\Omega)}+ 2\nu_\delta\left|\frac{\partial^2(I-P_{M,N})\mathbf{u}}{\partial z^2}\right|^2_{L^2(\Omega)}\leq C|\mathbf{F}|^2_{L^2}+C\left(K_2^2(t)+K_6^2(t)\right)|\partial_z\mathbf{u}|^2_{L^2}. \tag{3.37}$$

An application of the Gronwall inequality to (3.37) readily gives

$$|\partial_z\mathbf{u}(\cdot,t)|^2_{L^2(\Omega)}+\mu\int_0^t|\nabla\partial_z\mathbf{u}|^2_{L^2(\Omega)}\,\mathrm{d}s+ \nu\int_0^t|\frac{\partial^2\mathbf{u}}{\partial z^2}|^2_{L^2(\Omega)}\,\mathrm{d}s+2\mu_\delta\int_0^t|\nabla\partial_z(I-P_{M,N})\mathbf{u}|^2_{L^2(\Omega)}\,\mathrm{d}s+ 2\nu_\delta\int_0^t|\frac{\partial^2(I-P_{M,N})\mathbf{u}}{\partial z^2}|^2_{L^2(\Omega)}\,\mathrm{d}s\leq K_z(t), \tag{3.38}$$

with

$$K_z(t)=e^{C\left(K_2^2(t)+K_6^2(t)\right)t}\left(|\mathbf{u}_0|^2_{H^1}+C\int_0^t|\mathbf{F}|^2_{L^2}\,\mathrm{d}s\right).$$

To estimate $|\nabla\mathbf{u}|_{L^2}$
We multiply (3.12) by $-\Delta\mathbf{u}$ and integrate by parts over Ω,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{u}|^2_{L^2(\Omega)}+\int_\Omega\left((\mathbf{u}\cdot\nabla)\mathbf{u}+\left(\nabla\cdot\int_z^0\mathbf{u}\,\mathrm{d}\xi\right)\frac{\partial\mathbf{u}}{\partial z}\right)\cdot(-\Delta\mathbf{u})\,\mathrm{d}\Omega+ f\int_\Omega k\times\mathbf{u}\cdot(-\Delta\mathbf{u})\,\mathrm{d}\Omega+\frac{1}{\rho_0}\int_\Omega\nabla p_0\cdot(-\Delta\mathbf{u})\,\mathrm{d}\Omega+\mu|\Delta\mathbf{u}|^2_{L^2(\Omega)}+ \nu|\nabla\partial_z\mathbf{u}|^2_{L^2(\Omega)}+\mu_\delta|\Delta(I-P_{M,N})\mathbf{u}|^2_{L^2(\Omega)}+\nu_\delta|\nabla\partial_z(I-P_{M,N})\mathbf{u}|^2_{L^2(\Omega)} =(F,-\Delta\mathbf{u}) \tag{3.39}$$

We note that

$$\int_M(f\mathbf{k}\times\mathbf{u})\cdot\Delta\mathbf{u}\,\mathrm{d}\Omega=0,$$

and

$$\begin{aligned}\int_\Omega\nabla p_0\cdot(-\Delta\mathbf{u})\,\mathrm{d}\Omega&=-\int_M\nabla p_0\cdot\int_{-H}^0(\Delta\mathbf{u})\,\mathrm{d}z\,\mathrm{d}M\\&=-H\int_M\nabla p_0\cdot\Delta\overline{\mathbf{u}}\,\mathrm{d}M\\&=0.\end{aligned}$$

It is easy to see that

$$\int_\Omega \left((\mathbf{u}\cdot\nabla)\mathbf{u} + \left(\nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}\xi\right)\frac{\partial\mathbf{u}}{\partial z}\right)\cdot(\Delta\mathbf{u})\,\mathrm{d}\Omega \leq \\ C\int_\Omega \left(|\mathbf{u}||\nabla\mathbf{u}| + \int_{-H}^0 |\nabla\mathbf{u}|\,\mathrm{d}z|\partial_z\mathbf{u}|\right)|\Delta\mathbf{u}|\,\mathrm{d}\Omega.$$

Therefore we derive from (3.39) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{u}|^2_{L^2(\Omega)} + \frac{\mu}{2}|\Delta\mathbf{u}|^2_{L^2(\Omega)} + \nu|\nabla\partial_z\mathbf{u}|^2_{L^2(\Omega)} \\ +\mu_\delta|\Delta(I-P_{M,N})\mathbf{u}|^2_{L^2(\Omega)} + \nu_\delta|\nabla\partial_z(I-P_{M,N})\mathbf{u}|^2_{L^2(\Omega)} \leq \frac{1}{\mu}|\mathbf{F}|^2_{L^2} + \frac{\mu}{4}|\Delta\mathbf{u}|^2_{L^2} \\ + C\int_\Omega \left(|\mathbf{u}||\nabla\mathbf{u}| + \int_{-H}^0 |\nabla\mathbf{u}|\,\mathrm{d}z|\partial_z\mathbf{u}|\right)|\Delta\mathbf{u}|\,\mathrm{d}\Omega. \quad (3.40)$$

By the Young's inequality and the Sobolev interpolation inequality (A.4) for L^3 functions, we have

$$\int_\Omega |\mathbf{u}||\nabla\mathbf{u}||\Delta\mathbf{u}|\,\mathrm{d}\Omega \leq C|\mathbf{u}|_{L^6}|\nabla\mathbf{u}|_{L^3}|\Delta\mathbf{u}|_{L^2} \\ C \leq |\mathbf{u}|_{L^6}|\nabla\mathbf{u}|^{\frac{1}{2}}_{L^2}|\nabla\partial_z\mathbf{u}|^{\frac{1}{2}}_{L^2}|\Delta\mathbf{u}|_{L^2} + C|\mathbf{u}|_{L^6}|\nabla\mathbf{u}|^{\frac{1}{2}}_{L^2}|\Delta\mathbf{u}|^{\frac{3}{2}}_{L^2}.$$

We appeal to the following inequality,

$$\int_\Omega \left(\int_{-H}^0 |\nabla\mathbf{u}|\,\mathrm{d}z\right)|f||g|\,\mathrm{d}\Omega \leq C|f|_{L^2}|\mathbf{u}|^{\frac{1}{2}}_{H^1}|\mathbf{u}|^{\frac{1}{2}}_{H^2}|g|^{\frac{1}{2}}_{L^2}|g|^{\frac{1}{2}}_{H^1},$$

to obtain (by setting $f = \Delta\mathbf{u}$, $g = \partial_z\mathbf{u}$)

$$\int_\Omega \left(\int_{-H}^0 |\nabla\mathbf{u}|\,\mathrm{d}z\right)|\partial_z\mathbf{u}||\Delta\mathbf{u}|\,\mathrm{d}\Omega \leq C|\nabla\mathbf{u}|^{\frac{1}{2}}_{L^2}|\partial_z\mathbf{u}|^{\frac{1}{2}}_{L^2}|\nabla\partial_z\mathbf{u}|^{\frac{1}{2}}_{L^2}|\Delta\mathbf{u}|^{\frac{3}{2}}_{L^2}.$$

After these steps, we infer from (3.40) that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{u}|^2_{L^2} + \mu|\Delta\mathbf{u}|^2_{L^2} + \\ \nu|\nabla\partial_z\mathbf{u}|^2_{L^2} + 2\mu_\delta|\Delta(I-P_{M,N})\mathbf{u}|^2_{L^2} + 2\nu_\delta|\nabla\partial_z(I-P_{M,N})\mathbf{u}|^2_{L^2} \leq \\ \frac{4}{\mu}|\mathbf{F}|^2_{L^2} + C\left(|\mathbf{u}|^4_{L^6} + |\partial_z\mathbf{u}|^2_{L^2}|\nabla\partial_z\mathbf{u}|^2_{L^2}\right)|\nabla\mathbf{u}|^2_{L^2}. \quad (3.41)$$

We are ready to apply the Gronwall inequality, with use of (3.36), (3.38), to obtain

$$|\nabla \mathbf{u}(\cdot,t)|_{L^2}^2 + \mu \int_0^t |\Delta \mathbf{u}|_{L^2}^2 \, \mathrm{d}s + \nu \int_0^t |\nabla \partial_z \mathbf{u}|_{L^2}^2 \, \mathrm{d}s + 2\mu_\delta \int_0^t |\Delta (I - P_{M,N})\mathbf{u}|_{L^2}^2 \, \mathrm{d}s + 2\nu_\delta \int_0^t |\nabla \partial_z (I - P_{M,N})\mathbf{u}|_{L^2}^2 \, \mathrm{d}s \le K_V(t), \quad (3.42)$$

with

$$K_V(t) = e^{C\left(K_6^2 + K_2^2(t)\right)t + K_z^2(t)} \left(|\mathbf{u}_0|_{H^1}^2 + \frac{4}{\mu} \int_0^t |\mathbf{F}(\cdot,s)|_{L^2}^2 \, \mathrm{d}s \right).$$

Now that the key estimates are in place, with an argument similar to that for Theorem 2.1 in Section 2, we can show

Theorem 3.1. *For a given $T > 0$, let $\mathbf{F} \in L^2(0,T; L^2(\Omega))$, $\mathbf{u}_0 \in V$. Then there exists a unique strong solution of the system* (3.1)–(3.8) *which depends continuously on the initial data.*

The proof is omitted.

3.3. **Convergence of the solutions.** In this section we take $\mu_\delta \longrightarrow 0$, $\nu_\delta \longrightarrow 0$ in (3.12), and study the convergence of the solution of the system. We first rewrite (3.12) as follows:

$$\frac{\partial \mathbf{u}^\delta}{\partial t} + (\mathbf{u}^\delta \cdot \nabla)\mathbf{u}^\delta + (\nabla \cdot \int_z^0 \mathbf{u}^\delta \, \mathrm{d}\xi) \frac{\partial \mathbf{u}^\delta}{\partial z} + f\mathbf{k} \times \mathbf{u}^\delta + \frac{1}{\rho_0} \nabla p - \mu \Delta \mathbf{u}^\delta - \nu \frac{\partial^2 \mathbf{u}^\delta}{\partial z^2} - \mu_\delta \Delta (I - P_{M,N})\mathbf{u}^\delta - \nu_\delta \frac{\partial^2}{\partial z^2} (I - P_{M,N})\mathbf{u}^\delta = \mathbf{F}. \quad (3.43)$$

The superscript δ emphasizes the fact that the solution $\mathbf{u}^\delta$ depends on the spectral viscosity parameters μ_δ and ν_δ, which themselves are determined by the grid resolution. The proper relation between the parameters μ_δ and ν_δ and the grid resolution δ is the subject of a separate endeavor, and will be presented elsewhere. In this work we focus on the behavior of the solution $\mathbf{u}_\delta$ of system (3.43) as μ_δ and ν_δ tend to 0 (corresponding to the scenario when the grid becomes finer and finer).

Let $\mathbf{u}$ be the solution of the primitive equations without artificial viscosities, that is, $\mathbf{u}$ is the solution of

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + (\nabla\cdot\int_z^0 \mathbf{u}\,\mathrm{d}\xi)\frac{\partial \mathbf{u}}{\partial z} + f\mathbf{k}\times\mathbf{u} + \frac{1}{\rho_0}\nabla p - \mu\Delta\mathbf{u} - \nu\frac{\partial^2\mathbf{u}}{\partial z^2} = \mathbf{F}. \quad (3.44)$$

It can be shown that the system (3.44) plus (3.4)–(3.8) has unique global strong solutions, provided that the initial data and the forcing are sufficiently smooth. See [7, 20, 21] (These works use boundary conditions different from ours, but the case with periodic boundary conditions can be handled as well.)

To study the convergence of the solution of system (3.43), we subtract (3.44) from (3.43), let $\mathbf{v}^\delta = \mathbf{u}^\delta - \mathbf{u}$, and we have

$$\frac{\partial \mathbf{v}^\delta}{\partial t} + (\mathbf{u}^\delta\cdot\nabla)\mathbf{v}^\delta + (\nabla\cdot\int_z^0 \mathbf{u}^\delta\,\mathrm{d}\xi)\frac{\partial \mathbf{v}^\delta}{\partial z} + (\mathbf{v}^\delta\cdot\nabla)\mathbf{u} + (\nabla\cdot\int_z^0 \mathbf{v}^\delta\,\mathrm{d}\xi)\frac{\partial \mathbf{u}}{\partial z} + f\mathbf{k}\times\mathbf{v}^\delta - \mu\Delta\mathbf{v}^\delta - \nu\frac{\partial^2\mathbf{v}^\delta}{\partial z^2} - \mu_\delta\Delta(I-P_{M,N})\mathbf{u}^\delta - \nu_\delta\frac{\partial^2}{\partial z^2}(I-P_{M,N})\mathbf{u}^\delta = 0. \quad (3.45)$$

We take the inner product of (3.45) with $\mathbf{v}^\delta$, and integrate by parts over Ω to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{v}^\delta|_{L^2}^2 &+ \int_\Omega\left((\mathbf{u}^\delta\cdot\nabla)\mathbf{v}^\delta + (\nabla\cdot\int_z^0 \mathbf{u}^\delta\,\mathrm{d}\xi)\frac{\partial \mathbf{v}^\delta}{\partial z}\right)\cdot\mathbf{v}^\delta\,\mathrm{d}\Omega + \\ &\int_\Omega\left((\mathbf{v}^\delta\cdot\nabla)\mathbf{u} + (\nabla\cdot\int_z^0 \mathbf{v}^\delta\,\mathrm{d}\xi)\frac{\partial \mathbf{u}}{\partial z}\right)\cdot\mathbf{v}^\delta\,\mathrm{d}\Omega + \\ &\int_\Omega f\mathbf{k}\times\mathbf{v}^\delta\cdot\mathbf{v}^\delta\,\mathrm{d}\Omega + \mu|\nabla\mathbf{v}^\delta|^2 + \nu\left|\frac{\partial \mathbf{v}^\delta}{\partial z}\right|^2 \\ &= -\mu_\delta(\nabla(I-P_{M,N})\mathbf{u}^\delta,\, \nabla\mathbf{v}^\delta) - \nu_\delta\left(\frac{\partial}{\partial z}(I-P_{M,N})\mathbf{u},\, \frac{\partial}{\partial z}\mathbf{v}^\delta\right). \quad (3.46)\end{aligned}$$

We note that

$$\int_\Omega f\mathbf{k}\times\mathbf{v}^\delta\cdot\mathbf{v}^\delta\,\mathrm{d}\Omega = 0. \quad (3.47)$$

We can also verify by integration by parts, and using the boundary conditions (3.4)–(3.6) for $\mathbf{u}$, $\mathbf{u}^\delta$ (and therefore for $\mathbf{v}^\delta$), that

$$\int_\Omega\left((\mathbf{u}^\delta\cdot\nabla)\mathbf{v}^\delta + (\nabla\cdot\int_z^0 \mathbf{u}^\delta\,\mathrm{d}\xi)\frac{\partial \mathbf{v}^\delta}{\partial z}\right)\cdot\mathbf{v}^\delta\,\mathrm{d}\Omega = 0. \quad (3.48)$$

By Holder's inequality and the interpolation inequalities (A.4) and (A.5) in R^3, we find that

$$
\begin{aligned}
\left|\int_\Omega \big((\mathbf{v}^\delta\cdot\nabla)\mathbf{u}\big)\cdot\mathbf{v}^\delta\,\mathrm{d}\Omega\right| &\le C|\nabla\mathbf{u}|_{L^2}|\mathbf{v}^\delta|_{L^3}|\mathbf{v}^\delta|_{L^6},\\
\left|\int_\Omega \big((\mathbf{v}^\delta\cdot\nabla)\mathbf{u}\big)\cdot\mathbf{v}^\delta\,\mathrm{d}\Omega\right| &\le C|\nabla\mathbf{u}|_{L^2}|\mathbf{v}^\delta|_{L^2}^{\frac{1}{2}}|\nabla\mathbf{v}^\delta|_{L^2}^{\frac{3}{2}}. \qquad (3.49)
\end{aligned}
$$

Using Holder's inequality again we find that

$$
\begin{aligned}
&\left|\int_\Omega \left(\nabla\cdot\int_z^0 \mathbf{v}^\delta\,\mathrm{d}\xi\right)\frac{\partial\mathbf{u}}{\partial z}\cdot\mathbf{v}^\delta\,\mathrm{d}\Omega\right|\\
&\le C\int_M\left(\int_{-H}^0|\nabla\mathbf{v}^\delta|\,\mathrm{d}z\right)\left(\int_{-H}^0\left|\frac{\partial\mathbf{u}}{\partial z}\right||\mathbf{v}^\delta|\,\mathrm{d}z\right)\mathrm{d}x\,\mathrm{d}y\\
&\le C\int_M\left(\int_{-H}^0|\nabla\mathbf{v}^\delta|\,\mathrm{d}z\right)\left(\int_{-H}^0\left|\frac{\partial\mathbf{u}}{\partial z}\right|^2\mathrm{d}z\right)^{\frac{1}{2}}\left(\int_{-H}^0|\mathbf{v}^\delta|^2\,\mathrm{d}z\right)^{\frac{1}{2}}\mathrm{d}x\,\mathrm{d}y\\
&\le C\left(\int_M\left(\int_{-H}^0|\nabla\mathbf{v}^\delta|\,\mathrm{d}z\right)^2\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_M\left(\int_{-H}^0\left|\frac{\partial\mathbf{u}}{\partial z}\right|^2\mathrm{d}z\right)^2\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{4}}\times\\
&\quad\left(\int_M\left(\int_{-H}^0|\mathbf{v}^\delta|^2\,\mathrm{d}z\right)^2\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{4}}.
\end{aligned}
$$

We first note that

$$
\begin{aligned}
\left(\int_M\left(\int_{-H}^0|\nabla\mathbf{v}^\delta|\,\mathrm{d}z\right)^2\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} &\le C\left(\int_M\int_{-H}^0|\nabla\mathbf{v}^\delta|^2\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\\
&= C|\nabla\mathbf{v}^\delta|_{L^2(\Omega)}
\end{aligned}
$$

By the integral Minkowski inequality (A.6) and the interpolation inequality (A.2) in R^2, we find that

$$\begin{aligned}\left(\int_M\left(\int_{-H}^0\left|\frac{\partial\mathbf{u}}{\partial z}\right|^2\mathrm{d}z\right)^2\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} &\leq C\int_{-H}^0\left(\int_M\left|\frac{\partial\mathbf{u}}{\partial z}\right|^4\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}z\\ &\leq C\int_{-H}^0\left|\frac{\partial\mathbf{u}(\cdot,z)}{\partial z}\right|^2_{L^4(M)}\mathrm{d}z\\ &\leq C\int_{-H}^0\left|\frac{\partial\mathbf{u}(\cdot,z)}{\partial z}\right|_{L^2(M)}\left|\nabla\frac{\partial\mathbf{u}(\cdot,z)}{\partial z}\right|_{L^2(M)}\mathrm{d}z\\ &\leq C\left|\frac{\partial\mathbf{u}}{\partial z}\right|_{L^2(\Omega)}\left|\nabla\frac{\partial\mathbf{u}}{\partial z}\right|_{L^2(\Omega)}.\end{aligned}$$

Similarly, we have

$$\left(\int_M\left(\int_{-H}^0|\mathbf{v}^\delta|^2\,\mathrm{d}z\right)^2\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\leq C|\mathbf{v}^\delta|_{L^2}|\nabla\mathbf{v}^\delta|_{L^2}.$$

Therefore,

$$\left|\int_\Omega\left(\nabla\cdot\int_z^0\mathbf{v}^\delta\,\mathrm{d}\xi\right)\frac{\partial\mathbf{u}}{\partial z}\cdot\mathbf{v}^\delta\,\mathrm{d}\Omega\right|\leq C\left|\frac{\partial\mathbf{u}}{\partial z}\right|^{\frac{1}{2}}_{L^2(\Omega)}\left|\nabla\frac{\partial\mathbf{u}}{\partial z}\right|^{\frac{1}{2}}_{L^2(\Omega)}|\mathbf{v}^\delta|^{\frac{1}{2}}_{L^2}|\nabla\mathbf{v}^\delta|^{\frac{3}{2}}_{L^2}. \tag{3.50}$$

With (3.47)–(3.50), we derive from (3.46) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{v}^\delta|^2_{L^2}+\mu|\nabla\mathbf{v}^\delta|^2+\nu\left|\frac{\partial\mathbf{v}^\delta}{\partial z}\right|^2\leq\mu_\delta\left|\nabla(I-P_{M,N})\mathbf{u}^\delta\right|\left|\nabla\mathbf{v}^\delta\right|+\\ &\nu_\delta\left|\frac{\partial}{\partial z}(I-P_{M,N})\mathbf{u}\right|\left|\frac{\partial}{\partial z}\mathbf{v}^\delta\right|+C\left(\left|\frac{\partial\mathbf{u}}{\partial z}\right|^{\frac{1}{2}}_{L^2(\Omega)}\left|\nabla\frac{\partial\mathbf{u}}{\partial z}\right|^{\frac{1}{2}}_{L^2(\Omega)}\right)|\mathbf{v}^\delta|^{\frac{1}{2}}_{L^2}|\nabla\mathbf{v}^\delta|^{\frac{3}{2}}_{L^2}+\\ &\qquad C|\nabla\mathbf{u}|_{L^2}|\mathbf{v}^\delta|^{\frac{1}{2}}_{L^2}|\nabla\mathbf{v}^\delta|^{\frac{3}{2}}_{L^2}.\end{aligned} \tag{3.51}$$

By Young's inequality, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{v}^\delta|^2_{L^2}+\mu|\nabla\mathbf{v}^\delta|^2+\nu\left|\frac{\partial\mathbf{v}^\delta}{\partial z}\right|^2\leq 2\frac{\mu_\delta^2}{\nu}\left|\nabla(I-P_{M,N})\mathbf{u}^\delta\right|^2+\\ &\frac{\nu_\delta^2}{\nu}\left|\frac{\partial}{\partial z}(I-P_{M,N})\mathbf{u}\right|^2+C\left(\left|\frac{\partial\mathbf{u}}{\partial z}\right|^2_{L^2(\Omega)}\left|\nabla\frac{\partial\mathbf{u}}{\partial z}\right|^2_{L^2(\Omega)}+|\nabla\mathbf{u}|^4_{L^2}\right)|\mathbf{v}^\delta|^2_{L^2}.\end{aligned} \tag{3.52}$$

We notice that the a priori estimates obtained in the previous section are independent of μ_δ and ν_δ. We apply the Gronwall inequality to

(3.52), and, utilizing the a priori estimates (3.26), (3.38) and (3.42), we obtain

$$|\mathbf{v}^\delta(\cdot,t)|_{L^2}^2 + \int_0^t \mu |\nabla \mathbf{v}^\delta(\cdot,s)|^2 \,\mathrm{d}s + \nu \int_0^t \left| \frac{\partial \mathbf{v}^\delta}{\partial z}(\cdot,s) \right|^2 \mathrm{d}s$$
$$\leq e^{K_z^2(t)+K_V^2(t)} \left(\frac{2\mu_\delta}{\mu} + \frac{\nu_\delta}{\nu} \right) K_1(t). \quad (3.53)$$

We have just proved

Theorem 3.2. *Let $T > 0$ be given, and the other assumptions be the same as in Theorem 3.1. Then, as $\mu_\delta \,\&\, \nu_\delta \longrightarrow 0$,*

$$\mathbf{u}^\delta - \mathbf{u} \sim \sqrt{\mu_\delta + \nu_\delta} \longrightarrow 0 \quad \textit{in } L^\infty(0,T;L^2(\Omega)), \quad (3.54)$$
$$\mathbf{u}^\delta - \mathbf{u} \sim \sqrt{\mu_\delta + \nu_\delta} \longrightarrow 0 \quad \textit{in } L^2(0,T;H^1(\Omega)). \quad (3.55)$$

Appendix A. Some inequalities

We list here some functional inequalities that are frequently used in this paper.

L^p interpolation inequality

Let $\Omega \subset \mathbb{R}^3$, and $1 \leq p_1 \leq p \leq p_2$, $p_1 \neq p_2$. Let $u \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$. Then $u \in L^p(\Omega)$, and

$$||u||_{L^p} \leq ||u||_{L^{p_1}}^{s_1} ||u||_{L^{p_2}}^{s_2}, \quad (A.1)$$

with

$$s_1 = \frac{p_1}{p}\frac{p_2 - p}{p_2 - p_1}, \qquad s_2 = \frac{p_2}{p}\frac{p - p_1}{p_2 - p_1}.$$

Ladyzhenskaya/Sobolev inequalities in $\mathbb{R}^2$

Let $M \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundaries. For each $\phi \in H^1(M)$, the following inequalities hold:

$$||\phi||_{L^4(\Omega)} \leq C||\phi||_{L^2(\Omega)}^{\frac{1}{2}} ||\phi||_{H^1(\Omega)}^{\frac{1}{2}}, \quad (A.2)$$
$$||\phi||_{L^8(\Omega)} \leq C||\phi||_{L^6(\Omega)}^{\frac{3}{4}} ||\phi||_{H^1(\Omega)}^{\frac{1}{4}}. \quad (A.3)$$

Ladyzhenskaya/Sobolev inequalities in $\mathbb{R}^3$

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with piecewise smooth boundaries. For each $\phi \in H^1(\Omega)$, the following inequalities hold:

$$||\phi||_{L^3(\Omega)} \leq C||\phi||_{L^2(\Omega)}^{\frac{1}{2}} ||\phi||_{H^1(\Omega)}^{\frac{1}{2}}, \quad (A.4)$$
$$||\phi||_{L^6(\Omega)} \leq C||\phi||_{H^1(\Omega)}. \quad (A.5)$$

Minkowski integral inequality (for $p \geq 1$)
Let $\Omega_1 \subset \mathbb{R}^{m_1}$ and $\Omega_2 \subset \mathbb{R}^{m_2}$ be two measurable sets, with m_1 and m_2 being positive integers. Let $f(\xi, \eta)$ be a measurable function over $\Omega_1 \times \Omega_2$. Then

$$\left(\int_{\Omega_1} \left(\int_{\Omega_2} |f(\xi,\eta)|\, \mathrm{d}\eta\right)^p \mathrm{d}\xi\right)^{\frac{1}{p}} \leq \int_{\Omega_2} \left(\int_{\Omega_1} |f(\xi,\eta)|^p\, \mathrm{d}\xi\right)^{\frac{1}{p}} \mathrm{d}\eta. \quad \text{(A.6)}$$

The inequality (A.1) can be verified by the Holder's inequality. For (A.2)–(A.5) we refer to such classical texts as [1, 33]. A proof of (A.6) can be found in [16].

Acknowledgment

Q. Chen and M. Gunzburger are supported by the US Department of Energy grant number DE-SC0002624 as part of the *Climate Modeling: Simulating Climate at Regional Scale* program. This work is supported in part by grants from the National Science Foundation. Wang acknowledges helpful conversation with Ning Ju.

References

1. Robert A. Adams, *Sobolev spaces*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975, Pure and Applied Mathematics, Vol. 65. MR MR0450957 (56 #9247)
2. Joel Avrin and Chang Xiao, *Convergence of Galerkin solutions and continuous dependence on data in spectrally-hyperviscous models of 3D turbulent flow*, J. Differential Equations **247** (2009), no. 10, 2778–2798. MR MR2568157
3. L. C. Berselli, T. Iliescu, and W. J. Layton, *Mathematics of large eddy simulation of turbulent flows*, Scientific Computation, Springer-Verlag, Berlin, 2006. MR MR2185509 (2006h:76071)
4. Marcus Calhoun-Lopez and Max D. Gunzburger, *A finite element, multiresolution viscosity method for hyperbolic conservation laws*, SIAM J. Numer. Anal. **43** (2005), no. 5, 1988–2011 (electronic). MR MR2192328 (2007b:35223)
5. ———, *The efficient implementation of a finite element, multi-resolution viscosity method for hyperbolic conservation laws*, J. Comput. Phys. **225** (2007), no. 2, 1288–1313. MR MR2349182 (2008j:65157)
6. Chongsheng Cao and Edriss S. Titi, *Global well-posedness and finite-dimensional global attractor for a 3-D planetary geostrophic viscous model*, Comm. Pure Appl. Math. **56** (2003), no. 2, 198–233. MR MR1934620 (2003k:37129)
7. ———, *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, Ann. of Math. (2) **166** (2007), no. 1, 245–267. MR MR2342696
8. Q. Chen, J. Laminie, A. Rousseau, R. Temam, and J. Tribbia, *A 2.5D model for the equations of the ocean and the atmosphere*, Anal. Appl. (Singap.) **5** (2007), no. 3, 199–229. MR MR2340646 (2008h:35281)

9. Peter Constantin and Ciprian Foias, *Navier-Stokes equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988. MR MR972259 (90b:35190)
10. Daniel Castaño Díez, Max Gunzburger, and Angela Kunoth, *An adaptive wavelet viscosity method for hyperbolic conservation laws*, Numer. Methods Partial Differential Equations **24** (2008), no. 6, 1388–1404. MR MR2453940 (2009j:65242)
11. Jorgen S. Frederiksen, Martin R. Dix, and Steven M. Kepert, *Systematic energy errors and the tendency toward canonical equilibrium in atmospheric circulation models*, Journal of the Atmospheric Sciences **53** (1996), no. 6, 887–904.
12. A. E. Gill, *Atmosphere-ocean dynamics*, New York: Academic Press, 1982.
13. Jean-Luc Guermond and Serge Prudhomme, *Mathematical analysis of a spectral hyperviscosity LES model for the simulation of turbulent flows*, M2AN Math. Model. Numer. Anal. **37** (2003), no. 6, 893–908. MR MR2026401 (2004j:65150)
14. F. Guillén-González, N. Masmoudi, and M. A. Rodríguez-Bellido, *Anisotropic estimates and strong solutions of the primitive equations*, Differential Integral Equations **14** (2001), no. 11, 1381–1408. MR MR1859612 (2003b:76038)
15. Max Gunzburger, Eunjung Lee, Yuki Saka, Catalin Trenchea, and Xiaoming Wang, *Analysis of nonlinear spectral eddy–viscosity models of turbulence*, to appear.
16. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988, Reprint of the 1952 edition. MR MR944909 (89d:26016)
17. Changbing Hu, Roger Temam, and Mohammed Ziane, *The primitive equations on the large scale ocean under the small depth hypothesis*, Discrete Contin. Dyn. Syst. **9** (2003), no. 1, 97–131. MR MR1951315 (2003i:86003)
18. Ning Ju, *The global attractor for the solutions to the 3D viscous primitive equations*, Discrete Contin. Dyn. Syst. **17** (2007), no. 1, 159–179. MR MR2257424 (2008f:37177)
19. G-S. Karamanos and G. E. Karniadakis, *A spectral vanishing viscosity method for large-eddy simulations*, J. Comput. Phys. **163** (2000), no. 1, 22–50. MR MR1777720 (2001d:76069)
20. Georgy M. Kobelkov, *Existence of a solution "in the large" for ocean dynamics equations*, J. Math. Fluid Mech. **9** (2007), no. 4, 588–610. MR MR2374160
21. Igor Kukavica and Mohammed Ziane, *On the regularity of the primitive equations of the ocean*, Nonlinearity **20** (2007), no. 12, 2739–2753. MR MR2368323 (2008k:35379)
22. J.L. Lions, R. Temam, and S.H. Wang, *New formulations of the primitive equations of atmosphere and applications*, Nonlinearity **5** (1992), no. 2, 237–288. MR 93e:35088
23. ______, *On the equations of the large-scale ocean*, Nonlinearity **5** (1992), no. 5, 1007–1053. MR 93k:86004
24. Andrew J. Majda and Xiaoming Wang, *Non-linear dynamics and statistical theories for basic geophysical flows*, Cambridge University Press, Cambridge, 2006. MR MR2241372 (2009e:76214)
25. James C. McWilliams, *The emergence of isolated coherent vortices in turbulent flow*, Journal of Fluid Mechanics Digital Archive **146** (1984), no. -1, 21–43.

26. Haim Nessyahu and Eitan Tadmor, *The convergence rate of approximate solutions for nonlinear scalar conservation laws*, SIAM J. Numer. Anal. **29** (1992), no. 6, 1505–1519. MR MR1191133 (93j:65139)
27. J. Pedlosky, *Geophysical fluid dynamics, 2nd edition*, Springer, 1987.
28. M. Petcu, R. Temam, and M. Ziane, *Mathematical problems for the primitive equations with viscosity*, Handbook of Numerical Analysis. Special Issue on Some Mathematical Problems in Geophysical Fluid Dynamics (R. Temam P.G. Ciarlet EDs and J. Tribbia Guest Eds, eds.), Handb. Numer. Anal., Elsevier, New York, 2008.
29. A. Rousseau, R. Temam, and J. Tribbia, *Boundary conditions for an ocean related system with a small parameter*, Nonlinear PDEs and Related Analysis, vol. 371, Gui-Qiang Chen, George Gasper and Joseph J. Jerome Eds, Contemporary Mathematics, AMS, Providence, 2005, pp. 231–263.
30. ———, *The 3D primitive equations in the absence of viscosity: boundary conditions and well-posedness in the linearized case*, J. Math. Pures Appl. (9) **89** (2008), no. 3, 297–319. MR MR2401691
31. J. Smagorinsky, *General circulation experiments with the primitive equations. I. the basic experiment*, Monthly Weather Review **91** (1963), 99–152.
32. S. Stolz, P. Schlatter, and L. Kleiser, *High-pass filtered eddy-viscosity models for large-eddy simulations of transitional and turbulent flow*, Physics of Fluids **17** (2005), no. 6, 065103.
33. R. Temam, *Navier-Stokes equations*, AMS Chelsea Publishing, Providence, RI, 2001, Theory and numerical analysis, Reprint of the 1984 edition. MR MR1846644 (2002j:76001)